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Numerische Mechanik

Master Thesis

Investigation of nonholonomic characteristics in Free-floating space robot

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Abstract

Free-floating manipulators or robots in space which are attached to their satellite base, represent a nonholonomic mechanical system due to the conservation of angular momentum. This implies that planning of robotic end effector trajectories and their steering and control is highly complicated. This is more than ever valid if the manipulator degrees of freedom are increasing beyond the conventional number of six. Such kinematically redundant manipulators are often designed because of their flexibility in use and their increased skills.
It is known from several mechanical systems, specifically from moving vehicles, that nonholonomy can be favourably used in different ways. For example, there arises the possibility to steer the system using less number of inputs than is given by the generalized coordinates. This is not possible for holonomic systems that require as many inputs as the number of inde-pendent coordinates. The present work therefore shall investigate the characteristics arising from nonholonomy of a free-floating satellite equipped with a robot.
The investigations shall start with a literature survey of already existing work on nonholonomic systems. Then, specific emphasis shall be paid to the use of nonholonomy in satellite applications, again with focus on robots, specifically redundant robots, attached to their base. The work shall conclude with the set up of some simulation models of a free-floating satellite with a robot attached to it. It is suggested to run the simulations in a way to demonstrate the basic characteristics of the nonholonomic system,i.e., the linear and angular momentum of the system and the position of the system center of mass.

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Chapter 1

Introduction

Recently there has been a revival of interest in the study of mechanical systems with nonholonomic constraints in many applications ranging from Engineering to Robotics, wheeled vehicle and satellite dynamics, manipulation devices and locomotion system. This thesis report is intended for the beginners who wish to get the basic understating of the nonholonomic concepts from the point of view of differential geometry and nonlinear control theory as well. In mathematics, sub-Riemannian geometry is dealing with the nonholonomic systems.

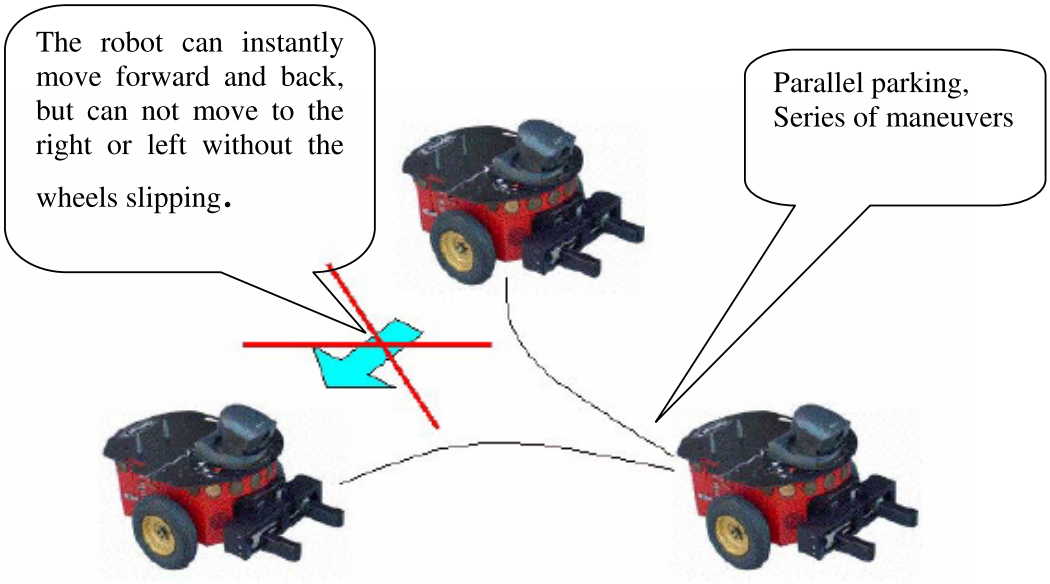

Figure 1.1: The wheeled robot (The nonholonomic mechanical system) [9]

1.1 Motivation

While a person walking on the surface, his or her velocity is not restricted on the plane that means he or she can instantly step to the right or left, as well as going forwards or backwards is known as an example of holonomic system. But in the case of nonholonomic systems, the allowable velocities of the systems are restricted by the nonholonomic constraints. The

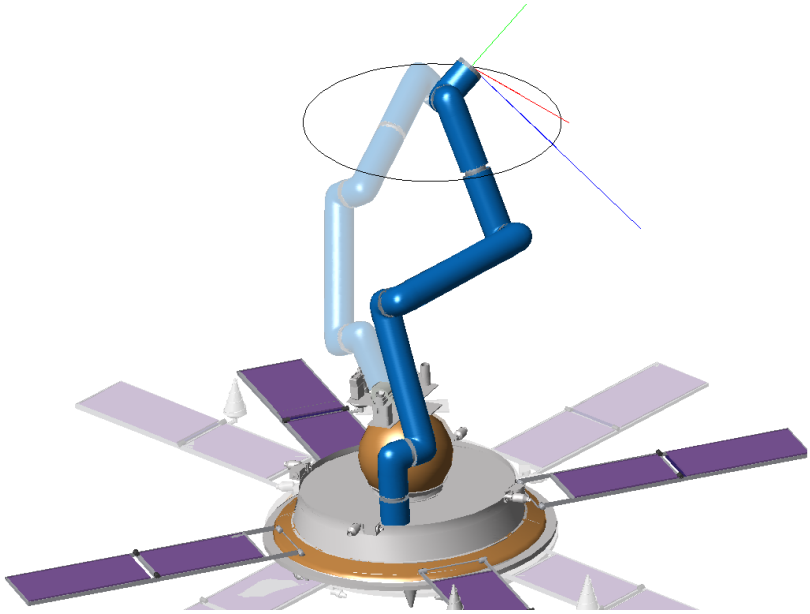

Figure 1.2: The Free-floating space robot [11]

wheeled robot is a simple example of a nonholonomic system and (kinematic) nonholonomic constraints are imposed on the system when it rolls without slipping over the floor i.e., the linear velocity of the point of contact with the surface is zero. It can move only in forwards and backwards directions, but not side to side (Figure 1). Figure 1 illustrates that though the internal configurations of the system return to its initital configurations but it does not gurantee return to the original system position.
Free-floating space robot (Figure 1.2) is another example of the nonholonomic system. In Free-floating space robot the (dynamic) nonholonomic constraints come due to the conservation of angular momentum by the basic Euler-Lagrange or Hamilton equations. Figure 1.2 shows the TECSAS scenario, where TECSAS stands for Technology Satellite for Demonstration and Verifi-cation of Space Systems. The TECSAS project is a cooperation between the Babakin Science and Research Space Center (Russia) and DLR. Here the redundant manipilator attached to a satellite base is considered as an entire system and compute the robotic joint and satellite base variables by applying the desired end effector trajectory in an inertial frame by means of a Lagrangian based optimization technique. The conservation law of linear and angular momentum are considered as an additional constraint besides the other kinematic constraints as well within the optimization algorithm [11].

1.2 Outlines of the thesis

Throughout this thesis report we discuss the nonholonomic criteria by means of differential geometry. We will now give a brief synopsis of the various chapters in this report.
In Chapter 2 we discuss the theoretical aspects of geometric and kinematic constraints of the mechanical systems and then introduce the holonomic and nonholonomic constraints and their degrees of freedom. The vertical rolling disk, a simple and famous example of nonholo-

nomic system is also shown in Chapter 2.
In Chapter 3 the integrability of the constraints by using classical mathematical approach is discussed and then we introduced the kinematic model into Pfaffian form and then derive the kinematic model based on non-linear control theory for further investigation of integrability of the constraints by means of Lie bracket operation.
In Chapter 4 we discuss some mathematical preliminaries from differential geometry such as Tangent Vector, Tangent Space, Vector field, Distribution and Lie bracket as well.
Chapter 5 we analyses the kinematic model by means of accessibility distribution to obtain the necessary condition to investigate whether the mechanical system is holonomic or non-holonomic.
In Chapter 6 we present the several kinematic models of nonholonomic systems including rolling disk, unicycle and car-like robot.
In Chapter 7 we derive the kinematic model of mobile redundant robot with one and two link manipulator and show that the redundancy under kinematic inversion is one of the source of nonholonomy.
In Chapter 8 we discuss the free-floating planar space robot of two and three bodies. The conservation of angular momentum gives the nonholonomic constraints. We derive the total angular momentum of the system and then convert into Pfaffian form to investigate the nonholonomic criteria.
Chapter 9 shows the same approach like Chapter 8 for the free-floating 3D space robot.
In all the simulation run we use MATLAB ODE45 as the primary numerical integration solver, if not mentioned otherwise.

Chapter 2

Mechanical System Constraints

Constraints are imposed on mechanical systems and the actual form of these constraints may be varied. For example, mechanical interconnections between the various bodies of the system or two separate bodies may be joined by a rod of fixed length. These constraints may impose restrictions on the possible geometrical positions of the system then they are called geometric constraint. Or may restrict the kinematical possible motions of the system, then they are defined as kinematic constraint. We can see that every geometric constraint must, at the same time, give rise to a certain kinematic constraint. However a constraint on the possible velocities of the system need not lead to restrict their possible positions.

2.1 Geometric and Kinematic Constraints

Let us consider a system without constraints then its geometrical position is determined by n quantities $q_1, q_2, \ldots, q_n$ known as generalized or independent coordinates. Then any arbitrary variation of these generalized coordinates in time corresponds to a certain motion of the unconstrained system. If the constraints are now imposed, there will be variations of the generalized coordinates $q_1, q_2, \ldots, q_n$ that no longer correspond to the motions of the system. The variations of the generalized coordinates and their values must now satisfy the constraints that are imposed on the system. The number of degrees of freedom equals the number of independent coordinates or generalized coordinates which are necessary to determine the configuration of the mechanical system. In general we can say the independent coordinates of a system of n degrees of freedom is

$$q_1,\ q_2, \ldots, q_n$$

Now, if we consider the motion of the mechanical system restricted by

$$h_i(q_1,\ q_2, \ldots, q_n) = 0, \qquad i = 1, \ldots, k < n, \tag{2.1}$$

we call these equations the geometric constraints.

On the other hand if we consider the motion of the mechanical system restricted by equations depending on both generalized coordinates and velocities,

$$a_i(q_1,\ q_2, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) = 0, \qquad i = 1, \ldots, k < n, \tag{2.2}$$

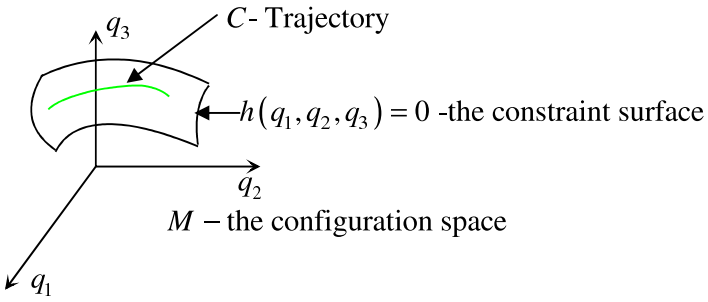

Figure 2.1: C Trajectory lies on the constraint surface

we call these equations the kinematic constraints. In mechanics such constraints are usually encountered in the **Pfaffian form**:

$$\mathbf{a}_i(\mathbf{q})\dot{\mathbf{q}} = 0, \qquad i = 1, \ldots, k < n, \tag{2.3}$$

or in compact form

$$\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = 0 \tag{2.4}$$

The kinematic constraints are integrable if the corresponding system of differential equations are integrable. Integrable kinematic constraints are essentially geometric constraints.

2.2 Holonomic and Nonholonomic Constraints

The concepts holonomic[1] and nonholonomic constraints were introduced by Hertz (1894) [6]. A constraint is called holonomic if it is geometric, or, in the case of kinematic constraints, if it is represented as an integrable differential form, i.e. if it can be expressed as an equation in the configuration variables; otherwise the constraint is called nonholonomic.

2.2.1 Holonomic Constraints and Degrees of Freedom

Let us consider a single particle in 3-dimensional configuration space M.

$$h_i(q_1,\, q_2, q_3, t) = 0, \qquad i = 1, \ldots, k < n, \tag{2.5}$$

If a holonomic constraint on the motion of the particle does not explicitly depend on time

$$h_i(q_1,\, q_2, q_3) = 0, \qquad i = 1, \ldots, k < n, \tag{2.6}$$

it is called scleronomic; otherwise it is called rheonomic, as in Eq. (2.5).
The constraint Eq. (2.6) is a 2-dimensional surface in the configuration space. As the motion of the system proceeds, a path is generated in this configuration space called a C trajectory (Figure 2.1) satisfying all constraints is a possible motion, but is not necessarily an actual motion because actual motions also obey Newton's Laws [5].

[1]Holonomic comes from the greek word and means "completely lawful" (in German: "ganz gesetzlich").

Degrees of Freedom
The number of linearly independent virtual displacements of a system is called the number of its degrees of freedom (DOF):

$$DOF = n - k, \qquad k = 1, \ldots, n \tag{2.7}$$

For a single particle if there are no constraints, $DOF = 3$, and it takes three independent parameters to specify the position of the system in configuration space. If there is one holonomic constraint,$DOF = 2$ that means two independent parameters are needed to specify the position of the system because motion is restricted on a surface. If there are two holonomic constraint, $DOF = 1$ that means one independent parameter needed to specify the position of the system because motion is restricted on a line. And if there are three holonomic constraint, $DOF = 0$, that means the particle is fixed.
So we understand that in holonomic systems, DOF may be reduced but DOF is equal to the number of generalized coordinates of the system. But this is not the case for a nonholonomic system.

2.2.2 Nonholonomic Constraints and Degrees of Freedom

The nonintegrable kinematic constraints that can not be reduced to geometric constraints is called nonholonomic constraints. The nonholonomic constraints may be found due to contact kinematics and/or due to conservation laws, for instance, conservation of angular momentum which sometimes is defined as dynamic constraints.
The presence of nonholonomic constraints limits the system mobility in a completely different way if compared to holonomic constraints. Let us consider a mechanical system with n generalized coordinates and k nonholonomic constraints. Then the generalized velocities at each point are confined to an *(n-k)* -dimensional subspace that means the mobility of the system is restricted to an *(n-k)* dimensional space. Hence, it is still possible to reach any points in its configuration space. Correspondingly, the number of degrees of freedom is reduced to *(n-k)* but we do not yield an explicit relation between the generalized coordinates.

2.3 The Vertical Rolling Disk

The vertical rolling disk is a basic and simple example of a system subject to nonholonomic constraints: a homogeneous disk is rolling without slipping on a horizontal plane. The configuration space of the vertical rolling disk is $M^2 = \mathbb{R}^2 \times S^1 \times S^1$ and parameterised by the generalized coordinates $\mathbf{q} = (x, y, \varphi, \theta)^T$ denoting the position coordinates *(x,y)* of the point of contact with the xy plane in a fixed frame. The angle θ is the rotation angle of the disk, and the angle φ characterizes the disk orientation with respect to x-axis as shown in Figure 2.2 [2].

[2] $\mathbb{R}^2 \times S^1$ means translation and rotation motion in the (x,y) plane and another S^1 means full rotation of the disk around its radial axis.

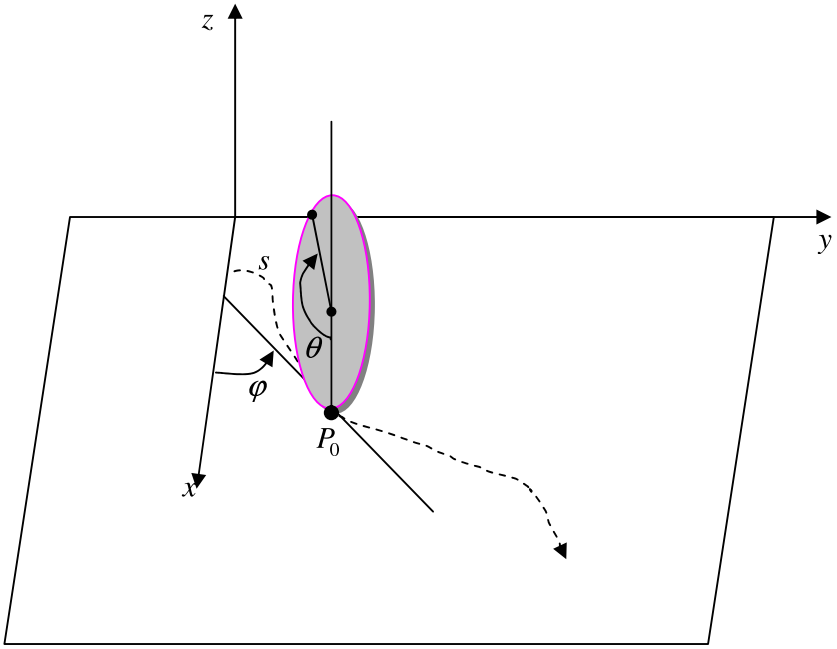

Figure 2.2: The geometry of the rolling disk($M = \mathbb{R}^2 \times S^1 \times S^1$)

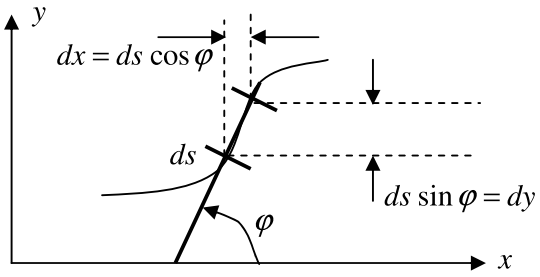

Figure 2.3: Geometrical quantities for the rolling constraint in the plane [10]

The relation between the two angles θ and φ and the position coordinates *(x,y)* of contact point P_0 can be derived by geometric considerations when we rotate the wheel about an infinitesimal small angle $d\theta$. With $dS = Rd\theta$ where R is the radius of the wheel, we get (Figure 2.3).

$$dx = ds\cos\varphi = Rd\theta\cos\varphi \tag{2.8}$$

$$dy = ds\sin\varphi = Rd\theta\sin\varphi \tag{2.9}$$

If we relate the infinitesimal small displacements dx and dy to an infinitesimal small time interval dt we get the velocities:

$$\dot{x} = \frac{dx}{dt} = R\frac{d\theta}{dt}\cos\varphi = R\dot{\theta}\cos\varphi \tag{2.10}$$

$$\dot{y} = \frac{dy}{dt} = R\frac{d\theta}{dt}\sin\varphi = R\dot{\theta}\sin\varphi \tag{2.11}$$

The above constraints can not be integrated. Because of the non-slipping condition, we get the nonholonomic constraints of the rolling disk:

$$\dot{x} = R\dot{\theta}\cos\varphi \tag{2.12}$$

$$\dot{y} = R\dot{\theta}\sin\varphi \tag{2.13}$$

We can write Eq. (2.12) and Eq. (2.13) into Pfaffian form

$$\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = 0 \tag{2.14}$$

where

$$\mathbf{A}(\mathbf{q}) = \begin{bmatrix} 1 & 0 & 0 & -R\cos\varphi \\ 0 & 1 & 0 & -R\sin\varphi \end{bmatrix} \tag{2.15}$$

and

$$\dot{\mathbf{q}} = \left(\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}\right) \tag{2.16}$$

Hence according to non-linear control system theory [1], at each configuration of $\mathbf{q}$, the admissible generalized velocities are those contained in the $m = (n-k) = 4-2 = 2$ dimensional null space (see Appendix B) of matrix $\mathbf{A}(\mathbf{q})$. Eventually if $\mathbf{G}(\mathbf{q}) = \mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q})$ is a basis for this space, all the feasible trajectories for the mechanical system are obtained as a solution of

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q})u_j = \mathbf{G}(\mathbf{q})\mathbf{u}, \qquad m = n-k \tag{2.17}$$

So we can write the following equation with

$$\mathbf{g}_1(\mathbf{q}) = \begin{bmatrix} R\cos\varphi \\ R\cos\varphi \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{g}_2(\mathbf{q}) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.18}$$

and from Eq. (2.17)

$$\dot{\mathbf{q}} = \mathbf{g}_1(\mathbf{q})u_1 + \mathbf{g}_2(\mathbf{q})u_2 \tag{2.19}$$

where

$$u = (u_1 \quad u_2)^T = (\dot{\theta} \quad \dot{\varphi})^T$$

is the vector of control variable of the system. In explicit writing Eq. (2.19) gives

$$\begin{aligned} \dot{q}_1 &= R\dot{\theta}\cos\varphi \\ \dot{q}_2 &= R\dot{\theta}\sin\varphi \\ \dot{q}_3 &= \dot{\varphi} \\ \dot{q}_4 &= \dot{\theta} \end{aligned}$$

This means that at each configuration (x, y, φ, θ) there are only two of the possible four directions available for the system to move. Note, however, that this does not mean that the configurations of the system are restricted.

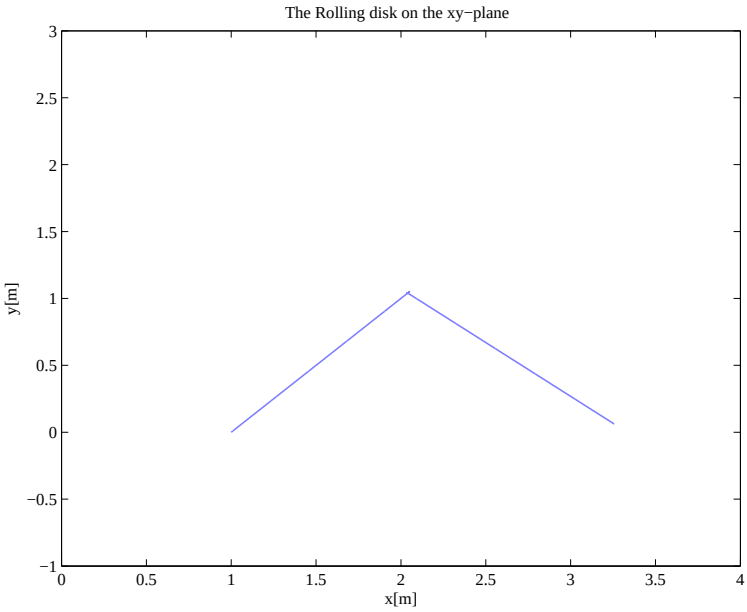

Figure 2.4: The rolling disk in $\mathbb{R}^2$ plane

2.3.1 Simulation results

In the following, some simulation results will be presented for understanding how control input is used to steer the nonholonomic systems. We assume that the initial state of our system Eq. (2.17) is $q_0 = (1, 0, \frac{\pi}{4}, 0)^T$, the control input $u = (u_1 \quad u_2)^T = (1 \quad 1)^T$ rad/sec and $R = 1\,m$

$$(u_1, u_2) = \begin{cases} (1,0), & 0 \leq t < \frac{T}{4} \quad \text{disk rolls (anticlockwise)} \\ (0,1), & \frac{T}{4} \leq t < \frac{T}{2} \quad \text{disk changes its orientation (anticlockwise)} \\ (-1,0), & \frac{T}{2} \leq t < \frac{3T}{4} \quad \text{disk rolls(clockwise)} \\ (0,-1), & \frac{3T}{4} \leq t < T \quad \text{disk changes its orientation (clockwise)} \end{cases}$$

to steer the system from $\mathbf{q}_0$ to $\mathbf{q}_f = (3.25, 0, \frac{\pi}{4}, 0)^T$, where $T = 2\pi$ sec i.e after one cycle $t = 2\pi$ sec of $\dot{\theta}$ and $\dot{\varphi}$.

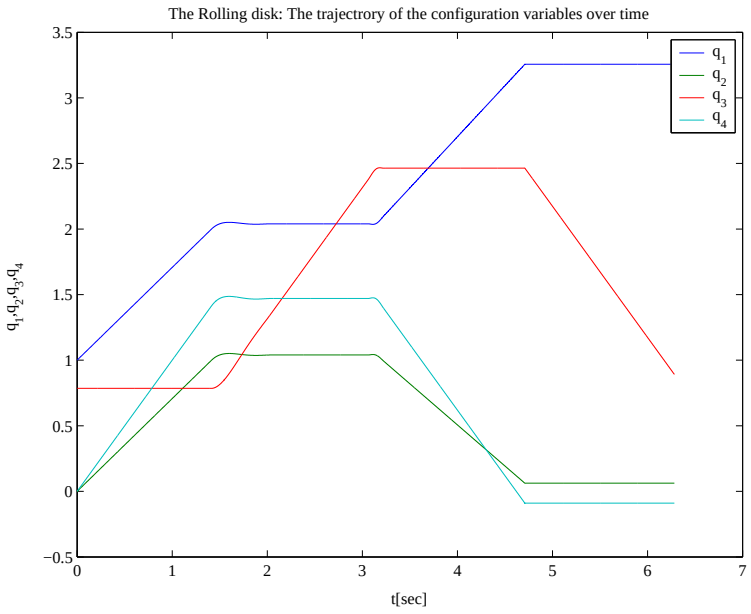

Figure 2.5: Trajectory of the configuration variables

The Figure 2.4 and Figure 2.5 show that at $t \geq 0$ the disk starts to roll and it rolls up to $t < \frac{T}{4}$ so the system reaches form $(1, 0, \frac{\pi}{4}, 0)$ to $\left(1 + \frac{\pi}{2\sqrt{2}}, \frac{\pi}{2\sqrt{2}}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ while $\varphi = \frac{\pi}{4}$ keeps constant. Within $\frac{T}{4} \leq t < \frac{T}{2}$ the disk changes its orientation only and it reaches at $(1 + \frac{\pi}{2\sqrt{2}}, \frac{\pi}{2\sqrt{2}}, \frac{3\pi}{4}, \frac{\pi}{2})$. After then within $\frac{T}{2} \leq t < \frac{3T}{4}$ the disk again rolls (clockwise) keeping $\varphi = \frac{3\pi}{4}$ constant and it reaches at $(3.25, 0, \frac{3\pi}{4}, 0)$. And finally within $\frac{3T}{4} \leq t < T$ the system reaches at $(3.25, 0, \frac{\pi}{4}, 0)$ while the disk changes its orientation only (clockwise) but other configuration variables keep constant. So we understand that there is no limitation on the configuration space M that may be attained by the vertical rolling disk by using appropriate control vector.

Special Remark on rolling disk

We found that Eq. (2.12) and Eq. (2.13) can not be integrated, but in a special case where the angle $\varphi = \varphi_0 =$ constant (that means the path is a straight line) the constraint can be integrated as follows:

$$\begin{cases} x - x_0 = R(\theta - \theta_0)\cos\varphi_0 \\ y - y_0 = R(\theta - \theta_0)\sin\varphi_0 \end{cases}$$

where θ_0, x_0 and y_0 define the initial conditions.
Now we have a unique relation between the angle θ and the two coordinates (x,y). This means that we have only one degree of freedom, the motion can be fully described by the angle θ So resetting the angle $\varphi = \varphi_0 =$ constant, changes the problem from a nonholonomic to a holonomic system [10].

Chapter 3

Integrability of the Constraints

In the previous example of the vertical rolling disk we found that there are two constraints and both of them are nonintegrable that means the system is completely nonholonomic in nature. In this chapter we will elaborate more about the integrability of the system constraints and its nonholonomic characteristics.

3.1 Determination of Holonomic/Nonholonomic Constraints

In section 2.1 we already mentioned that we can write the constraints in the **Pfaffian form**. If the **Pfaffian form** is an exact differential, then the constraint is integrable. The following theorem is a very useful result.

Theorem

Let us consider a single Pfaffian constraint

$$\sum_{j=1}^{m} \mathbf{a}_j(\mathbf{q})\dot{\mathbf{q}}_j = 0 \tag{3.1}$$

Then it is necessary and sufficient for the existence of an integral of this equation of the form

$$f(q_1,\, q_2, \ldots, q_n) = 0 \tag{3.2}$$

that the equations

$$a_\gamma \left(\frac{\partial a_\beta}{\partial q_\alpha} - \frac{\partial a_\alpha}{\partial q_\beta}\right) + a_\beta \left(\frac{\partial a_\alpha}{\partial q_\gamma} - \frac{\partial a_\gamma}{\partial q_\alpha}\right) + a_\alpha \left(\frac{\partial a_\gamma}{\partial q_\beta} - \frac{\partial a_\beta}{\partial q_\gamma}\right) = 0, \tag{3.3}$$

$\alpha, \beta, \gamma = 1, 2, \ldots, n$ be simultaneously and identically satisfied [5].

Example 1. For the following differential constraint in $\mathbb{R}^3$ [1]

$$\dot{q}_1 + \dot{q}_1\dot{q}_2 + \dot{q}_3 = 0 \tag{3.4}$$

we can write the above equation in **Pfaffian form**

$$\begin{bmatrix} 1 & q_1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = 0 \tag{3.5}$$

with $\mathbf{a}(\mathbf{q}) = (a_1, a_2, a_3) = (1, q_1, 1)$. Thus for Eq.(3.4) to be integrable the only requirement is that

$$a_1 \left(\frac{\partial a_2}{\partial q_3} - \frac{\partial a_3}{\partial q_2} \right) + a_2 \left(\frac{\partial a_3}{\partial q_1} - \frac{\partial a_1}{\partial q_3} \right) + a_3 \left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1} \right) = 0 \tag{3.6}$$

and then we obtain

$$a_1(0-0) + a_2(0-0) + a_3 \left(0 - \frac{\partial a_2}{\partial q_1} \right) = -a_3 \neq 0 \tag{3.7}$$

Hence the above equation is not integrable.

Example 2. Consider the following two differential constraints in $\mathbb{R}^3$ [1]

$$\dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 = 0 \tag{3.8}$$
$$\dot{q}_1 + \dot{q}_2 + q_1\dot{q}_3 = 0 \tag{3.9}$$

When regarding each of the two equations separately (the first one is the previous example and second one is similar to the first one) we find immediately that neither one is integrable. But when taken together, we can get the following

$$\begin{aligned} \dot{q}_1 + (q_1 + 1)\dot{q}_2 &= 0 \\ \dot{q}_1 + (q_1 + 1)\dot{q}_3 &= 0 \end{aligned} \tag{3.10}$$

These above equations are integrable and we obtain

$$\begin{aligned} q_2 + log(q_1 + 1) &= c_1 \\ q_2 - q_3 &= c_2 \end{aligned} \tag{3.11}$$

where c_1 and c_2 are constants.

So when we are dealing with multiple kinematic constraints, the nonholonomy of each constraint considered separately is not sufficient to infer that the whole set of constraints is nonholonomic. In fact, it may still happen that $k_h \leq k$ independent linear combinations of the constraints are integrable. So we may conclude that the system configurations are restricted to the $(n - k_h)$-dimensional subspace by the level surfaces of $f_i(\mathbf{q})$ as follows

$$\{\mathbf{g}_1 \in \mathbb{R}^n : f_1(\mathbf{q}) = c_1, \ldots, f_{n-k_h}(\mathbf{q}) = c_{n-k_h}\}$$

In the particular case of $k_h = k$, the set of differential constraints is completely equivalent to a set of holonomic constraints; hence it is itself holonomic.

3.2 Integrability and Kinematic Model

Consider a collection of mechanical particles in n-dimensional configuration space whose position vector is

$$\mathbf{r} = \mathbf{r}(q_1,\, q_2, \ldots, q_n) \tag{3.12}$$

Then the velocity vector $\dot{\mathbf{r}}$ is a tangent vector, since

$$\dot{\mathbf{r}} = \frac{d}{dt}\mathbf{r}(q_1,\, q_2, \ldots, q_n) = \sum_{1=1}^{n} \frac{\partial \mathbf{r}}{\partial q_i}\dot{q}_i \tag{3.13}$$

If there exists one holonomic constraint in the following form

$$\begin{aligned} q_n &= aq_1 + bq_2 + \ldots + q_{n-1} \\ &= f(q_1,\, q_2, \ldots, q_{n-1}) \end{aligned} \tag{3.14}$$

then we can write the position vector in the following way

$$\begin{aligned} \mathbf{r} &= \mathbf{r}(q_1,\, q_2, \ldots, q_{n-1}, f(q_1,\, q_2, \ldots, q_{n-1})) \\ &= \mathbf{r}(q_1,\, q_2, \ldots, q_{n-1}) \end{aligned} \tag{3.15}$$

That means the dimension of the configuration space is reduced by one holonomic constraints. But if we consider one nonholonomic constraints ($k_{nh} = 1$) in the following form

$$\begin{aligned} \dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 + \ldots + \dot{q}_n &= 0 \\ f(\dot{\mathbf{q}}, \mathbf{q}) &= 0 \\ \mathbf{A}(\mathbf{q})\dot{\mathbf{q}} &= 0 \end{aligned} \tag{3.16}$$

Then the instantaneous mobility of the system is restricted to an $(n-1)$ dimensional space but the position vector is the same as before,

$$\mathbf{r} = \mathbf{r}(q_1,\, q_2, \ldots, q_n) \tag{3.17}$$

which implies that the degrees of freedom are reduced by nonholonomic constraints. But the system is able to reach any configuration space with specific motion sequence.
To find out these motions we will consider the nonlinear control theory to formulate the associated kinematic model.
If we consider k Pfaffian constraints as given by Eq.(2.4), at each configuration $\mathbf{q}$, the admissible generalized velocities are those contained in the $(n-k)$-dimensional null space of $\mathbf{A}(\mathbf{q})$.

If $\mathbf{G}(\mathbf{q}) = \mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q})$ is a basis for this space, all the feasible trajectories for the mechanical system are obtained as a solution of

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q})u_j = \mathbf{G}(\mathbf{q})\mathbf{u}, \qquad m = n - k \tag{3.18}$$

for any arbitrary $\mathbf{u}(t)$. Eq.(3.18) is regarded as a nonlinear control system with state vector $\mathbf{q} \in \mathbb{R}^n$ and control input $\mathbf{u} \in \mathbb{R}^m$. From a mechanical point of view, it is an underactuated system, because there are less number of inputs than generalized coordinates ($n > m$).

The choice of $\mathbf{G}(\mathbf{q})$ in Eq. (3.18) is not unique, and we can choose the columns $\mathbf{g}_j$ in such a way that the corresponding u_j has direct physical meaning. Moreover, there is no relationship with the true mechanical input with vector $\mathbf{u}$. For this reason, Eq. (3.16) is referred to as the kinematic model of the constraint system. And the vector fields $\mathbf{g}(j)$ have a contribution for further analysis of the system to become a holonomic or nonholonomic, which will be discussed in the next Chapter.

Chapter 4

Lie Bracket and Nonholonomy

In the previous Chapter we came to know the kinematic model of the system as well as the vector fields **g**. In this chapter we would like to show how these vector fields **g** act as an important role to investigate the system behaviour being either holonomic or nonholonomic. Since there is a close link between nonholonomic constraints and controllability of the nonlinear systems. The analysis of nonlinear control systems requires the concept from differential geometry. For this reason in this chapter we will also discuss some mathematical concept from differential geometry such as Tangent Vector, Tangent Space, Vector field, Distribution and Lie Bracket as well. Detailed Lie bracket calculations are shown in Appendix C.

4.1 Tangent Vector, Tangent Space and Vector Field

Tangent Vector is an operation that maps smooth real function on neighborhood of $\boldsymbol{q}$ to the real line (Figure 4.1)[8].

Tangent Space is the set of all tangent vector at point $\mathbf{q} \in M$ (Figure 4.2)[7].

A vector field is an operator that associates a tangent vector X_q with $\mathbf{q}$. The vector fields give the direction in which the system can move at a given point $\mathbf{q}$ on the configuration space M (Figure 4.3)[8].

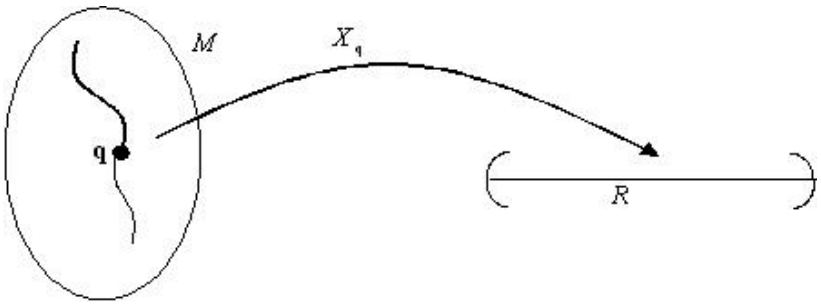

Figure 4.1: Tangent vector

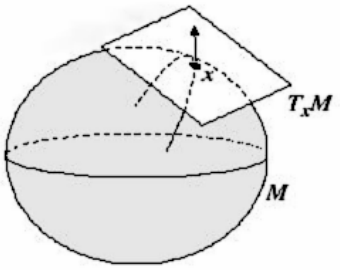

Figure 4.2: Tangent Space

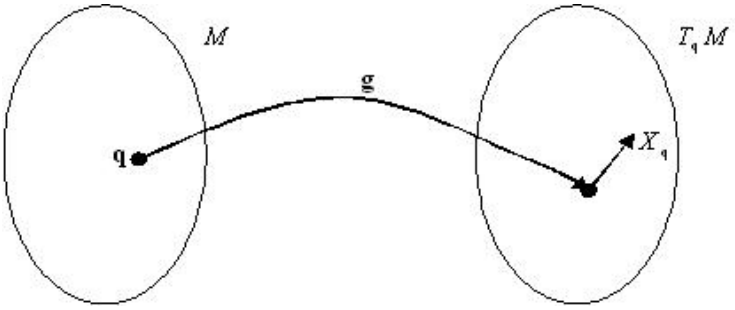

Figure 4.3: Vector field

An integral curve of a vector field is a smooth curve mapping (t_1, t_2) to the configuration variables in the configuration space which satisfy $\dot{c}(t) = g(c(t))$ (Figure 4.4) where $c(t)$ is considered as the flow $\phi_t^g(\mathbf{q})$ of the vector field $\mathbf{g}$ [8].
For example in the linear systems $\dot{\mathbf{q}} = \mathbf{g}(\mathbf{q}) = \mathbf{A}\mathbf{q}$ where the flow $\phi_t^g(\mathbf{q}) = e^{\mathbf{A}t}$.

The flow has a very important property. Let us allow a pebble to flow into a vector field for time t_1, and then flow a little more for time t_2. The result will be the same if the pebble had flowed for time $t_1 + t_2$. Then we get the following composition of their flows [4].

$$\phi_{t1}^{g_1} \circ \phi_{t2}^{g_1} = \phi_{t1+t2}^{g_2}$$

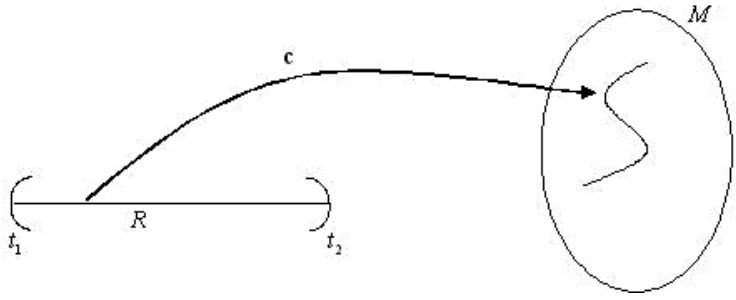

Figure 4.4: Integral curve

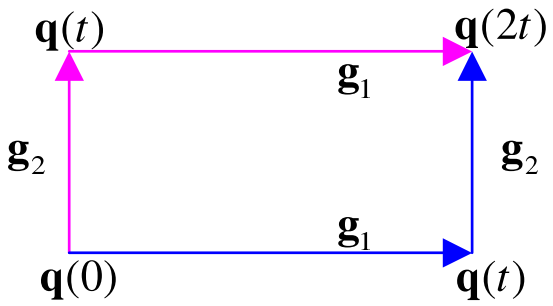

Figure 4.5: The closed parallelogram (The commute flows)

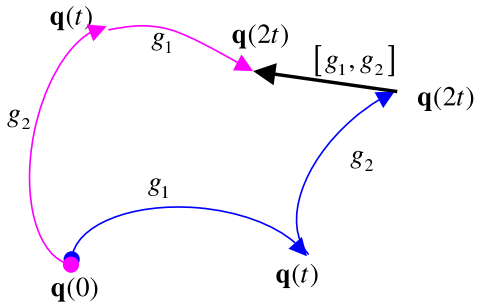

Figure 4.6: The non-commutative flows

This means that

$$\phi_{-t}^{g_1} \circ \phi_t^{g_1} = \phi_0^{g_2} = \phi_t^{g_1} \circ \phi_{-t}^{g_1}$$

and $\phi_t^{g_1}$ has a smooth inverse, so it is what we call a diffeomorphism, an isomorphism in the world of smooth manifolds. A smooth manifold is a type of topological space that locally like some $\mathbb{R}^n$, in such a way that we can do calculus on it and the adjective 'smooth' means that we can take as many derivatives as we want [4].
Now we consider the two constant vector fields $\mathbf{g}_1$ and $\mathbf{g}_2$. Then the composition of their flows is generally commutative, that means if we drop the pebble in order to flow with $\mathbf{g}_1$ for time t, and then with $\mathbf{g}_2$ for time t, the pebble will end up the same place if it flows first with $\mathbf{g}_2$ and then with $\mathbf{g}_1$ (Figure 4.5). If the vector fields are not constant then the composition of their flows is generally non-commutative (Figure 4.6) and that is

$$\phi_t^{g_1} \circ \phi_t^{g_2} = \phi_t^{g_2} \circ \phi_t^{g_1}$$

We can define another new vector field $[\mathbf{g}_1, \mathbf{g}_2]$ or $[\mathbf{g}_2, \mathbf{g}_1]$ to find out how much the flows fail to commute. And that is

$$[\mathbf{g}_1, \mathbf{g}_2] = -[\mathbf{g}_2, \mathbf{g}_1] \tag{4.1}$$

called Lie bracket of $\mathbf{g}_1$ and $\mathbf{g}_2$.

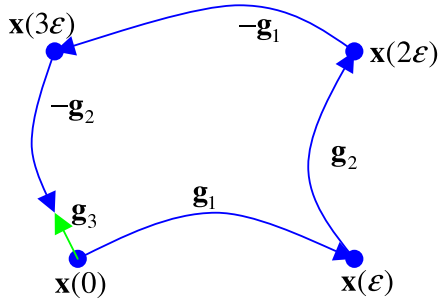

Figure 4.7: The Lie bracket motion

4.2 Physical Interpretation of Lie Bracket

Consider the two vector fields g_1 and g_2 their associate differential equation is

$$\dot{\mathbf{x}} = \mathbf{g}_1(\mathbf{x})u_1 + \mathbf{g}_2(\mathbf{x})u_2 \tag{4.2}$$

where u_1 and u_2 are the control inputs. If these two inputs are never active at the same time, the solution of the Eq. (4.2) is obtained by composing the flows relative to $\mathbf{g}_1$ and $\mathbf{g}_2$. Considering the following input sequence:

$$(u_1, u_2) = \begin{cases} (1,0), & 0 \leq t < \varepsilon \\ (0,1), & \varepsilon \leq t < 2\varepsilon \\ (-1,0), & 2\varepsilon \leq t < 3\varepsilon \\ (0,-1), & 3\varepsilon \leq t < 4\varepsilon \end{cases} \tag{4.3}$$

where ϵ is an infinitesimal interval of time. If the trajectory arising from Eq. (4.3) does not form a loop i.e., it end up to the starting point, then the motion primitives do not commute. Therefore, sequence of motion primitives in Eq. (4.3) will be referred as the commutator motion. The solution of the differential equation at time 4ε is obtained by following the flow $\mathbf{g}_1$, then $\mathbf{g}_2$, then $-\mathbf{g}_1$, then finally $-\mathbf{g}_2$ as follows (Figure 4.7) [1]:

$$\mathbf{x}(4\varepsilon) = \phi_{\varepsilon}^{-g_2} \circ \phi_{\varepsilon}^{-g_1} \circ \phi_{\varepsilon}^{g_2} \circ \phi_{\varepsilon}^{g_1}(\mathbf{x}(0)) \tag{4.4}$$

We will get the solution of Eq. (4.4) by using Taylor series, and we will get the trajectory from the first motion primitive,$(u_1, u_2) = (1, 0)$, that can be expressed as

$$\mathbf{x}(\varepsilon) = \mathbf{x}(0) + \varepsilon\dot{\mathbf{x}}(0) + \frac{1}{2}\varepsilon^2\ddot{\mathbf{x}}(0) + \dots \tag{4.5}$$

where

$$\ddot{\mathbf{x}} = \frac{d}{dt}\left(\dot{\mathbf{x}} = \mathbf{g}_1(\mathbf{x})\right) = \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\mathbf{g}_1(\mathbf{x}) \tag{4.6}$$

Putting $\ddot{x}$ into Eq. (4.5)

$$\mathbf{x}(\varepsilon) = \mathbf{x}(0) + \varepsilon\dot{\mathbf{x}}(0) + \frac{1}{2}\varepsilon^2\left(\frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\right)_{x(0)}\mathbf{g}_1(\mathbf{x}(0)) + \dots \tag{4.7}$$

Then for the second motion primitive $(u_1, u_2) = (0, 1)$ the Taylor series expansion is

$$\mathbf{x}(2\varepsilon) = \mathbf{x}(\varepsilon) + \varepsilon \mathbf{g}_2(\mathbf{x}(\varepsilon)) + \frac{1}{2}\varepsilon^2 \left(\frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\right)_{x(\varepsilon)} \mathbf{g}_2(\mathbf{x}(0)) + \dots \tag{4.8}$$

To simplify the expression, the evalution at $\mathbf{x}(0)$ has been dropped from every occurrence of $\mathbf{g}_1$ and $\mathbf{g}_2$ and their derivatives. By using the Taylor series expansion in Eq. (4.7) we can obtain the expression for $g_2(\mathbf{x}(\varepsilon))$ to express $\mathbf{x}(\varepsilon)$. After then substitute the first term of Eq. (4.8) by using Eq. (4.7). Then Eq. (4.8) becomes

$$\mathbf{x}(2\varepsilon) = \mathbf{x}(0) + \varepsilon \mathbf{g}_2 + \varepsilon^2 \left(\frac{1}{2}\frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\mathbf{g}_1 + \frac{\partial \mathbf{g}_2}{\partial \mathbf{x}}\mathbf{g}_1 + \frac{1}{2}\frac{\partial \mathbf{g}_2}{\partial \mathbf{x}}\mathbf{g}_2\right) + \dots \tag{4.9}$$

The Taylor series expansion for the third primitive motion

$$\mathbf{x}(3\varepsilon) = \mathbf{x}(0) + \varepsilon \mathbf{g}_2 + \varepsilon^2 \left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{x}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\mathbf{g}_2 + \frac{1}{2}\frac{\partial \mathbf{g}_2}{\partial \mathbf{x}}\mathbf{g}_2\right) + \dots \tag{4.10}$$

Finally, after all four motion primitives we get the following Taylor series expansion

$$\mathbf{x}(4\varepsilon) = \mathbf{x}(0) + \varepsilon^2 \left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{x}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\mathbf{g}_2\right) + O(\varepsilon^3) \tag{4.11}$$

In Eq. (4.11) the coordinate-dependent expression is

$$[\mathbf{g}_1, \mathbf{g}_2] = \left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{x}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\mathbf{g}_2\right) = \mathbf{g}_3 \tag{4.12}$$

is called the Lie bracket of $\mathbf{g}_1$ and $\mathbf{g}_2$. The above computation shows that, at each point, infinitesimal motion is possible not only in the directions of $\mathbf{g}_1$ and $\mathbf{g}_2$, but also in the directions of their Lie bracket, $\mathbf{g}_3$ [1] . If the commutator motion Eq. (4.3) can not produce any net motion i.e. $\mathbf{g}_3 = 0$, then we get $\mathbf{x}(4\varepsilon) = \mathbf{x}(0)$ that simply means g_1 and g_2 commute (Figure 4.5). In Figure 4.5 we consider the flow $\mathbf{g}_1$, then $\mathbf{g}_2$, then $-\mathbf{g}_1$, then finally $-\mathbf{g}_2$ then at $\mathbf{x}(4\varepsilon)$the system reaches its starting point. Then we can conclude that if the vector fields are commuting that means we are trapped on a surface. Then it is impossible to leave the surface by using the commutator motion, and if the system is trapped on a surface, the system is completely integrable. We can form a vector space from the vector fields of the system that is called the system distribution (more details in section 4.3) and if we have commuting vector fields then we get the corresponding distribution known as involutive distribution. Then we come up with another conclusion that if we get the involutive distribution then we can say the system is completely integrable (Frobenius Theorem, Appendix D) .

4.2.1 Lie Bracket Example: Unicycle Model

In the case of the unicycle model (Figure 4.8) [3], we will get the following kinematic model

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \nu \cos\theta \\ \nu \sin\theta \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \nu + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega = \mathbf{g}_1 u_1 + \mathbf{g}_2 u_2 \tag{4.13}$$

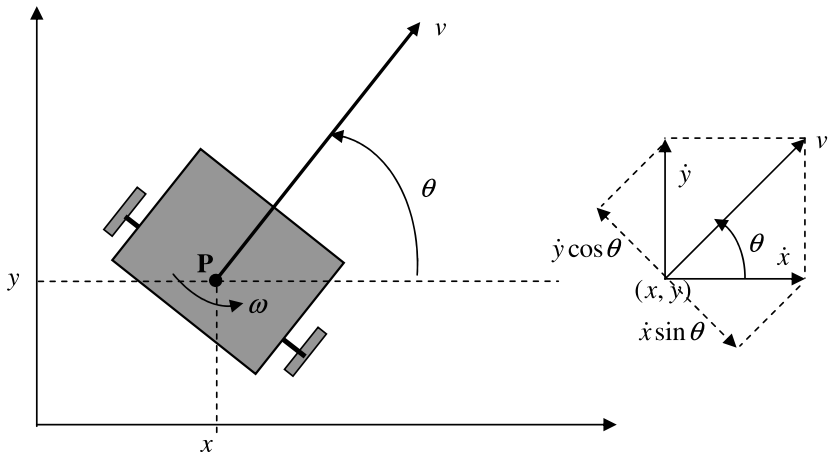

Figure 4.8: The unicycle ($\mathbb{R}^2 \times S^1$) (The two wheels collapse into one single wheel located at mid point)

Table 4.1: Lie Bracket examples for different $\mathbf{g}$ vectors

Case	$\mathbf{q}_0$	$\mathbf{u} = (u_1 \quad u_2)^T$ rad/sec	Active vector field	Remark
1	$\left(0, 0, \frac{\pi}{4}\right)^T$	$(1 \quad 0)^T$	$\mathbf{g}_1$	Pure translation (Figure 4.9)
2	$\left(0, 0, \frac{\pi}{4}\right)^T$	$(0 \quad 1)^T$	$\mathbf{g}_2$	Pure rotation (Figure 4.10)
3	$\left(0, 0, \frac{\pi}{4}\right)^T$	$(1 \quad 1)^T$	$\mathbf{g}_1, \mathbf{g}_2$	Combined motion (Figure 4.11)
4	$\left(0, 0, \frac{\pi}{4}\right)^T$	$u_3 = 1$	$\mathbf{g}_3$	Sideways motion (Figure 4.12)

where

$$\mathbf{u} = (u_1 \quad u_2)^T = (\nu \quad \omega)^T$$

and ν is the translational velocity, and ω is the angular velocity.

In this example we will get the two vector fields $\mathbf{g}_1$ and $\mathbf{g}_2$ and the physical meaning of $\mathbf{g}_1$ is pure translation and $\mathbf{g}_2$ is pure rotation of the motion. We will discuss the same problem in details in section 6.2.

Now we will present some simulation results for different cases which are shown in Table 4.1.

In case 1 we take the input $u_1 = 1$ rad/sec and $u_2 = 0$ so that only the vector field $\mathbf{g}_1$ is actively playing on the system. Figure 4.9 shows only the translation motion of the system while q_3 keeps constant.

In case 2 we take the input $u_1 = 0$ and $u_2 = 1$ rad/sec so that only the vector field $\mathbf{g}_2$ is actively playing on the system. Figure 4.10 shows only the rotational motion of the system while q_1,q_2 keep constant.

In case 3 we take the input $u_1 = 1$ rad/sec and $u_2 = 1$ rad/sec so that the both vector fields are actively playing on the system. Figure 4.11 shows the combined motion of the system.

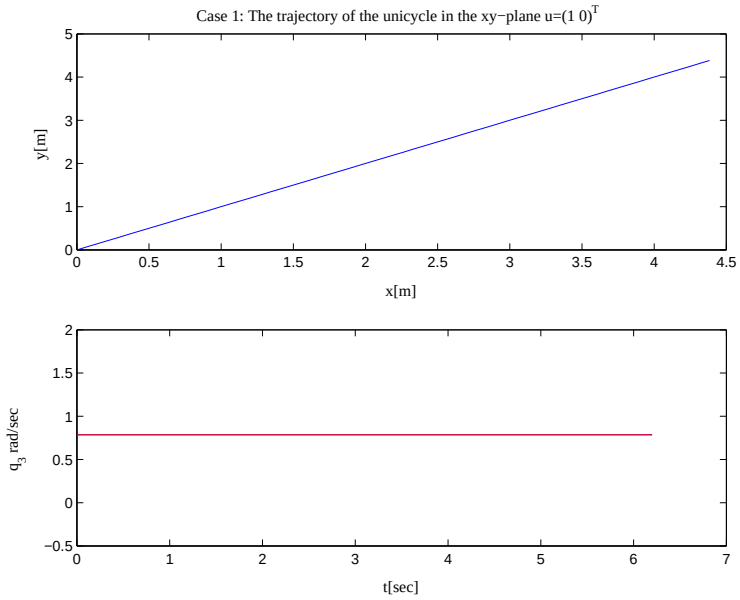

Figure 4.9: Pure translation

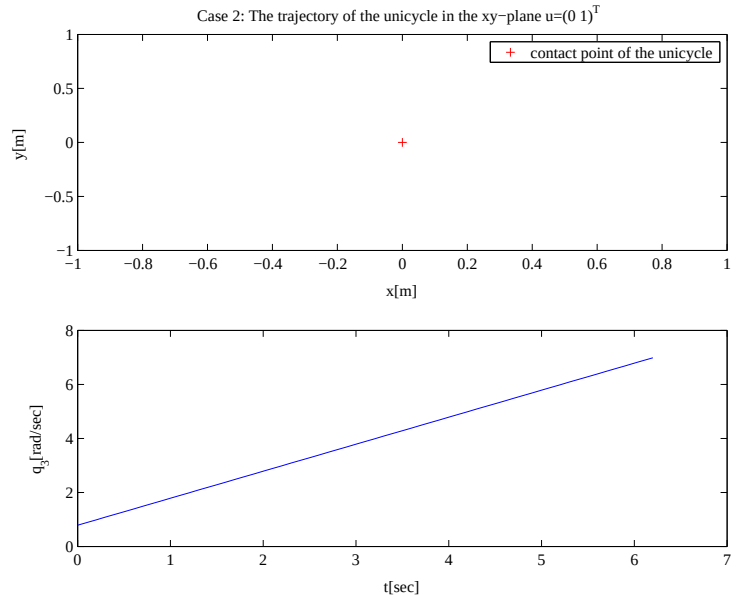

Figure 4.10: Pure rotation

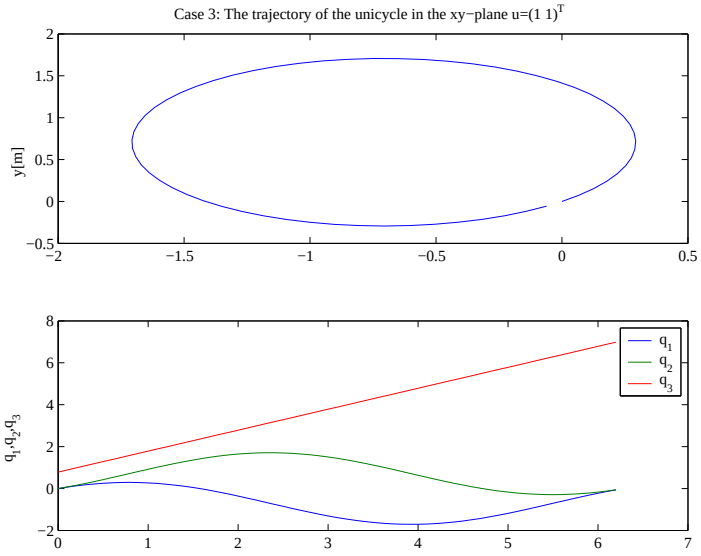

Figure 4.11: The combined motion

We know that by applying Lie bracket operation we can get other vector fields $\mathbf{g}$ to determine all possible directions of the motion.
If $[\mathbf{g}_1, \mathbf{g}_2]$ is not in the span of vector fields $\mathbf{g}_1$ and $\mathbf{g}_2$ then by concatenating flows of $\mathbf{g}_1$ and $\mathbf{g}_1$, we obtain the motion in a new independent direction (see Appendix C):

$$\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2] = [\sin\theta \quad -\cos\theta \quad 0]^T$$

A set of vector fields $\mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{g}_3$ is independent everywhere. After applying the Lie Bracket we find n linearly independent columns, then we can control our system in all n variables. In the unicycle example the combinations of $\mathbf{g}_1$ and $\mathbf{g}_2$ directions are shown in Figure 4.11 while the motion $[\mathbf{g}_1, \mathbf{g}_2]$ takes the unicycle sideways and corresponds to the parking manoeuvre (Figure 4.12).
As we discussed before in previous section 4.1 we can easily get the sideways motion with the following input profile:

$$(u_1, u_2) = \begin{cases} (1,0), & 0 \le t < \frac{1}{4}2\pi \\ (0,1), & 2\pi \le t < \frac{2}{4}2\pi \\ (-1,0), & \frac{2}{4}2\pi \le t < 3\frac{2\pi}{4} \\ (0,-1), & 3\frac{2\pi}{4} \le t < 42\pi \end{cases} \tag{4.14}$$

In Figure 4.13 we find that at the first primitive motion the unicycle translates (forward) along the vector field $\mathbf{g}_1$ while $\mathbf{q}_3$ keeping constant. Then at the second primitive motion the unicycle rotates (anticlockwise) along the vector field $\mathbf{g}_2$ while $\mathbf{q}_1$, $\mathbf{q}_2$ keeping constant. Then at the third primitive motion the unicycle translates (backward) along the vector field $\mathbf{g}_1$ while $\mathbf{q}_3$ keeping constant. Then finally at the fourth primitive motion the unicycle rotates (clockwise) along the vector field $\mathbf{g}_2$.

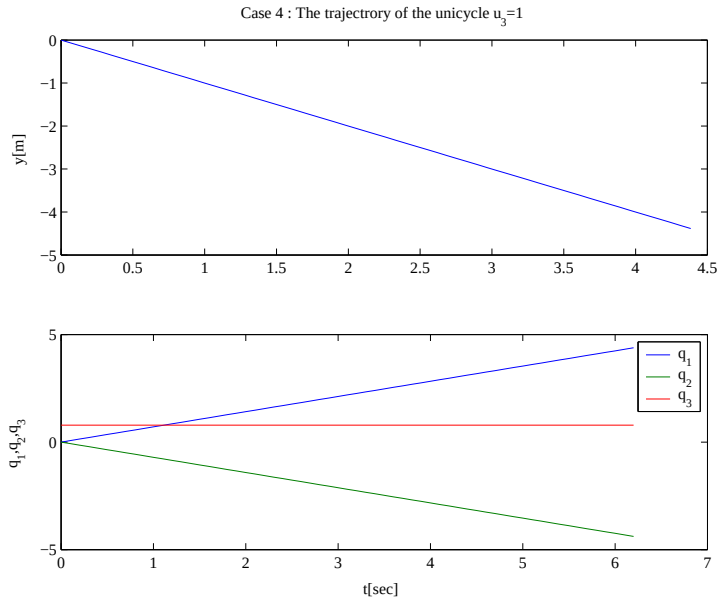

Figure 4.12: The unicycle sideways motion

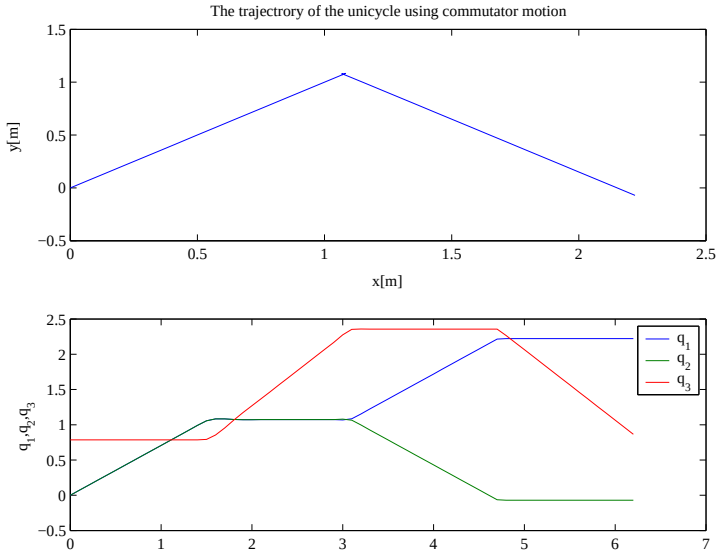

Figure 4.13: The unicycle sideways motion using commutator motion.

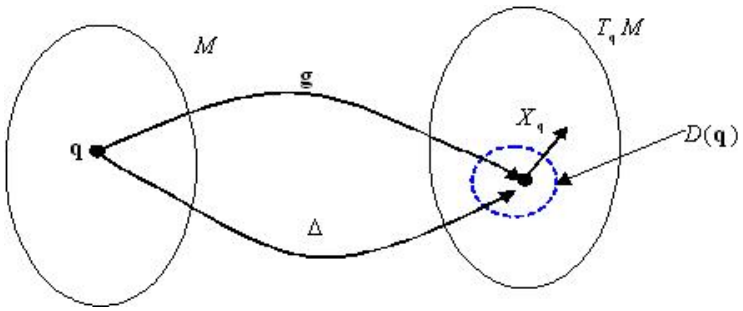

Figure 4.14: Distribution

4.3 Distribution and Nonholonomy

4.3.1 Distribution

It is an operator that maps the point $\mathbf{q}$ on the configuration space M to a linear subspace $D(\mathbf{q})$ of the tangent space $T_{\mathbf{q}}M$ Figure 4.14 [8].
So if we have the vector fields $\mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q})$ and if we assume that these vector fields are linearly independent then we can get the following m-dimensional vector space with $u_i \in \mathbb{R}^m$

$$\dot{\mathbf{q}} = \mathbf{g}_1(\mathbf{q})u_1 + \mathbf{g}_2(\mathbf{q})u_2 + \ldots + \mathbf{g}_m(\mathbf{q})u_m$$

which is called the distribution of the set $\{\mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q})\}$ of vector fields, and it is denoted as Δ . Since the distribution is almost a vector space, the span notation from linear algebra often is used to define it:

$$\begin{aligned}\Delta(\mathbf{q}) &= \text{span}\{\mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q})\} \\ &= \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q})u_j \\ &= \mathbf{G}(\mathbf{q})\mathbf{u} \in D(\mathbf{q}) \subset T_qM\end{aligned}$$

The dimension of a vector space is the number of independent basis vectors. Therefore, the dimension of $\Delta(\mathbf{q})$ is the rank of $\mathbf{G}(\mathbf{q})$ and it is denoted by dim $(\Delta(\mathbf{q}))$. A distribution $\Delta(\mathbf{q})$ is involutive if it is closed under Lie Bracket operation :

$$\mathbf{g}_1(\mathbf{q}) \in D(\mathbf{q}) \quad \text{and} \quad \mathbf{g}_2(\mathbf{q}) \in D(\mathbf{q})$$

$$[\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q})] \in D(\mathbf{q})$$

If for all $\mathbf{q} \in \mathbb{R}^n$, the $\dim(\Delta(\mathbf{q})) = r$ and further if r = constant for all $\mathbf{q}$, then we call r the dimension of the distribution. If for all $\mathbf{q} \in \mathbb{R}^n$, the $\dim(\Delta(\mathbf{q})) = m$ then the $\Delta(\mathbf{q})$ is called a non-singular or regular distribution. The points $\mathbf{q} \in \mathbb{R}^n$ for which the $\dim(\Delta(\mathbf{q})) < m$ are called the singular points and the corresponding distribution is called the singular distribution. A distribution $\Delta(\mathbf{q})$ is completely integrable if there exist $(n-k)$

real-valued functions h_i such that

$$\frac{\partial h_i}{\partial \mathbf{q}}\mathbf{g}_j(\mathbf{q}) = 0, \mathbf{q} \in \mathbb{R}^n, \mathbf{g}_j(\mathbf{q}) \in \Delta \tag{4.15}$$

for all $i = 1, \ldots, n-k$ and $j = 1, \ldots, k$. In this case we will get the level surfaces

$$\{\mathbf{g}_1 \in \mathbb{R}^n : h_1(\mathbf{q}) = c_1, \ldots, h_{n-k}(\mathbf{q}) = c_{n-k}\}$$

which is called the integral manifold of $\Delta(\mathbf{q})$.

4.3.2 Nonholonomy

If we have non-singular distribution then there exists another set of $\mathbf{g}$ vectors which we will get from Lie bracket operation. From the kinematic model we know that the dimension of $\mathbf{G}(\mathbf{q})$ is $(n \times m)$. Then $\mathbf{G}(\mathbf{q})$ can be easily augmented into another distribution matrix, $\mathbf{G}'(\mathbf{q})$, which is called accessibility distribution (more details in section 5.2) matrix of $n \times (m + k)$ dimension, by adding $[\mathbf{g}_i \quad \mathbf{g}_j]$ as a new column where k is the number of nonholonomic constraints.
Consider $\mathbf{G}'(\mathbf{q})$ has full rank this simply means we will get linearly independent $\mathbf{g}_i$ vectors which implies that the system is completely nonholonomic, otherwise it is holonomic or partial nonholonomic. In the case of the **unicycle model**, we have $n = 3$ and

$$\Delta_1 = \text{span}\{\mathbf{g}_1, \mathbf{g}_2\}$$

$$\dim(\Delta_1) = 2$$

which means that $\mathbf{G}(\mathbf{q})$ is non-singular. And we get

$$\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2]^T = [\sin\theta \quad -\cos\theta \quad 0]^T \in \Delta_1 \neq \text{involutive}$$

$$\Delta_2 = \text{span}\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\} \text{ and } \Delta_2 \text{ is the involute closure of } \Delta_1.$$

Then we get the following accessibility distribution matrix:

$$\mathbf{G}'(\mathbf{q}) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{bmatrix} \tag{4.16}$$

So the rank of $\mathbf{G}'(\mathbf{q}) = 3$ which means full rank. And the determinant of $\mathbf{G}'(\mathbf{q})$ is non-zero for all $\mathbf{q} \in \mathbb{R}^3$, which means that $\mathbf{g}_3$ is linearly independent of $\mathbf{g}_1$ and $\mathbf{g}_2$. So by **Frobenius theorem** (see Appendix D) the system is not integrable i.e. the system is nonholonomic.

Special Remark

If we have only one vector field in the kinematic model (Eq. 3.18) i.e., $m = 1$ then its corresponding distribution is always involute, hence the system is integrable. In section 8.1, the two-body Planar Space Robot, we will get only one vector field.

Chapter 5

Classification of Mechanical Systems

We already know from Chapter 3 that by using Lie bracket we can get all independent vector fields $\mathbf{g}$ which determine the possible directions of the motion. These vector fields $\mathbf{g}$ further are related to the motion planning problem of the nonholonomic systems. In this Chapter we will classify the nonholonomic system based on these independent vector fields $\mathbf{g}$ and their corresponding distribution which is called accessibility distribution.

5.1 Controllability and Accessible Region

In Chapter 2 and Chapter 3 we have discussed the vertical rolling disk and unicycle. Both of these cases we found that we can steer them from $\mathbf{q}_0$ to $\mathbf{q}_1$ by using proper input vector fields $\mathbf{u}$. That means there exists a trajectory $\mathbf{q}(t)$ from $\mathbf{q}_0$ to $\mathbf{q}_1$ that satisfies their own (nonholonomic) kinematic constraints. For this reason both of these (nonholonomic) systems are controllable and there is no restriction to access the whole configuration space M (The dimension of the accessible region is n).

But if we have holonomic kinematic constraints then the motion of the corresponding system is always confined on the constraint surface or the system motion is confined to an m-dimensional manifold (The dimension of the accessible region is m).

So we can conclude the kinematic constraints imply a loss of accessibility of the system configuration space M. Now at this moment we can define the number of kinematic constraints by

$$k = k_h + k_{nh} \tag{5.1}$$

where

$$k_h = \text{number of holonomic kinematic constraints}$$

and

$$k_{nh} = \text{number of nonholonomic kinematic constraints}$$

If $k_{nh} = 0$ the accessibility loss is maximal. Hence the integrability of the kinematic constraints define the dimension of the accessible region of the configuration space.

5.2 Accessibility Distribution

The dimension of the accessible region of the configuration space is determined by the accessibility distribution. For this reason recall the associate kinematic model of Pfaffian constraints (Eq. 3.18):

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q}) u_j = \mathbf{G}(\mathbf{q})\mathbf{u}, \qquad m = n - k \tag{5.2}$$

where

n number of generalized co-ordinates
k number of kinematic constraints
m mobility of the system

Here the dimension of $\mathbf{G}(\mathbf{q})$ is $(n \times m)$ and we can write down the distribution of $\mathbf{G}(\mathbf{q})$ as follows

$$\Delta_{\mathbf{G}(\mathbf{q})} = \text{span}\{\mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q})\} \tag{5.3}$$

Considering is a non-singular or regular distribution, means has full rank. Then we can compute repeated Lie brackets (Eq. (4.12), see Appendix C) of the input vector fields $\mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q})$ of the system (Eq. (5.2)) to get other vector fields g to determine all possible directions of motion. If we have constraints then we will get the following linearly independent vector fields g by applying the Lie bracket operation:

$${}_L\mathbf{G}(\mathbf{q}) = \{{}_L\mathbf{g}_1(\mathbf{q}), \ldots, {}_L\,\mathbf{g}_{k_{nh}}(\mathbf{q})\} \tag{5.4}$$

And we can write down the corresponding distribution as follows

$$\Delta_L\mathbf{G}(\mathbf{q}) = \text{span}\{{}_L\mathbf{g}_1(\mathbf{q}), \ldots, {}_L\,\mathbf{g}_{k_{nh}}(\mathbf{q})\} \tag{5.5}$$

After then we can form $\mathbf{G}'(\mathbf{q})$ matrix consisting of $\mathbf{G}(\mathbf{q})$ and ${}_L\mathbf{G}(\mathbf{q})$ This new matrix is called the accessibility distribution matrix of dimension $n \times (m + k_{nh})$. That means $\mathbf{G}(\mathbf{q})$ can be easily augmented into another distribution matrix, $\mathbf{G}'(\mathbf{q})$ by adding k_{nh} new columns, shown by the following Eq (5.6).

$$\mathbf{G}'(\mathbf{q}) = \{\mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q}), {}_L\,\mathbf{g}_1(\mathbf{q}), \ldots, {}_L\,\mathbf{g}_{k_{nh}}(\mathbf{q})\} \tag{5.6}$$

So we can define the accessibility distribution of the system

$$\Delta_{\mathbf{G}'(\mathbf{q})} = \text{span}\{\mathbf{g}_1(\mathbf{q}), \ldots, \mathbf{g}_m(\mathbf{q}), {}_L\,\mathbf{g}_1(\mathbf{q}), \ldots, {}_L\,\mathbf{g}_{k_{nh}}(\mathbf{q})\} \tag{5.7}$$

which is the involutive closure distribution[1] of the nonsingular distribution $\Delta_{\mathbf{G}(\mathbf{q})}$, that means it spans $\mathbb{R}^{m+k_{nh}}$. Then we obtain $\dim(\Delta_{\mathbf{G}'(\mathbf{q})}) = m + k_{nh}$ so we will get the following conclusion (Figure 5.1):

[1]The involute closure $\overline{\Delta}$ of a distribution Δ is its closure under the Lie bracket operation.

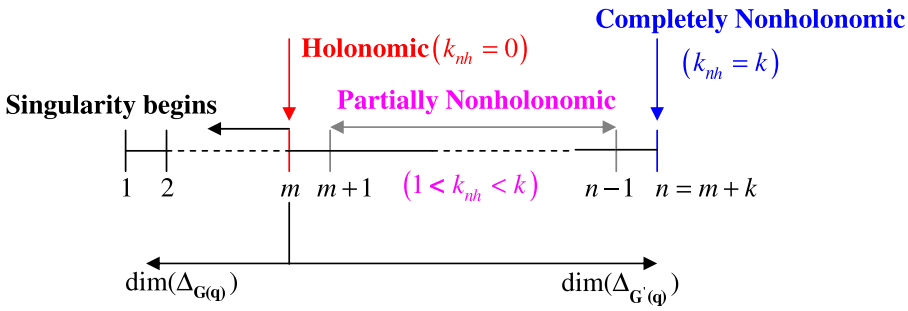

Figure 5.1: Classification of Mechanical System

if k_{nh}=k, max dim($\Delta_{\mathbf{G'(q)}}$) $= m+k=n$	System is completely nonholonomic
if $k_{nh}=0$,max dim($\Delta_{\mathbf{G'(q)}}$) $= m+0=n$	System is completely holonomic
$m+1 \leq$ dim($\Delta_{\mathbf{G'(q)}}$) $< n$	System is partially nonholonomic
$m+1 <$ dim($\Delta_{\mathbf{G(q)}}$)	Singular distribution,
	Singular point exists in $q \in \mathbb{R}^n$
	Physically this means loss of one or more DOF's

5.3 Summary

To investigate the mechanical system based on Lie bracket operation we will do the following procedure.
1^{st} step: Write down the kinematic constraints of the mechanical system into Pfaffian form:

$$\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = 0$$

2^{nd} step: Write down the kinematic model of the mechanical system by means of non-linear control theory:

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q})u_j = \mathbf{G}(\mathbf{q})\mathbf{u}, \qquad m = n-k$$

3^{rd} step: To determine the rank of $\mathbf{G}_{n\times m}(\mathbf{q})$ The necessary condition for further investigation whether the system is holonomic or nonholonomic is:

$$\text{rank}\mathbf{G}(\mathbf{q}) \geq m$$

Then we can create other linearly independent vector fields $\mathbf{g}$ by means of Lie bracket operation. We can get the number of maximum linearly independent vector fields $\mathbf{g}$ is equal to the number of generalized coordinate system.
4^{th} step: To generate the accessibility distribution matrix $\mathbf{G'}(\mathbf{q})$, by augmenting the $\mathbf{G}_{n\times m}(\mathbf{q})$ by adding $[\mathbf{g}_i, \mathbf{g}_j]$ as a new column.

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{G'}(\mathbf{q})\mathbf{u'} \\ &= \Delta_{\mathbf{G'(q)}}\end{aligned}$$

5^{th} step: To investigate by means of **Frobenius Theorem** [1]. To determine the rank of the accessibility distribution matrix $\mathbf{G}'(\mathbf{q})$ If the $\mathbf{G}'(\mathbf{q})$ has full rank, i.e.,

$$\operatorname{rank} \mathbf{G}'(\mathbf{q}) = n$$

and

$$\det \mathbf{G}'(\mathbf{q}) \neq 0$$

then according to the **Frobenius Theorem**, the corresponding system is fully nonholonomic.

Chapter 6

Examples of the Kinematic Models

In this Chapter 6 and in the three succeeding Chapters 7, 8 and 9 we will discuss and investigate the nonholonomic nature of various mechanical systems. Hereby, we will take care of the results obtained by the formulation for classification of nonholonomy derived in the Chapters before. We will consider three different sources of nonholonomy:

A. Rolling contacts without slipping (Chapter 6)

B. Mobile redundant robots with given EE trajectory (Chapter 7) (without taking care of the linear and angular momentum of the system)

C. Conservation of angular momentum in multibody systems (Chapters 8 and 9)

In this Chapter 6 we therefore will investigate the following three examples:

1. The Vertical Rolling Disk
2. Unicycle.
3. Car-like Mobile Robot.

6.1 Vertical Rolling Disk

For this system, the configuration space has dimension $n = 4$ (see Figure 2.2) and $\mathbf{q} = (x, y, \theta, \varphi)^T$ The input vector fields of kinematic model corresponding to Eq. (2.14) is computed as

$$\mathbf{g}_1 = [R\cos\varphi \quad R\sin\varphi \quad 0 \quad 1]^T \tag{6.1}$$

$$\mathbf{g}_2 = [0 \quad 0 \quad 1 \quad 0]^T \tag{6.2}$$

So we obtain the regular distribution $\Delta_{\mathbf{G}(\mathbf{q})} = \text{span}\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q})\}$ Then compute (see Appendix C)

$$\mathbf{g}_3 = [\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q})] = [R\sin\varphi \quad -R\cos\varphi \quad 0 \quad 0]^T \tag{6.3}$$

$$\mathbf{g}_4 = [\mathbf{g}_3(\mathbf{q}), \mathbf{g}_4(\mathbf{q})] = [R\sin\varphi \quad R\cos\varphi \quad 0 \quad 0]^T \tag{6.4}$$

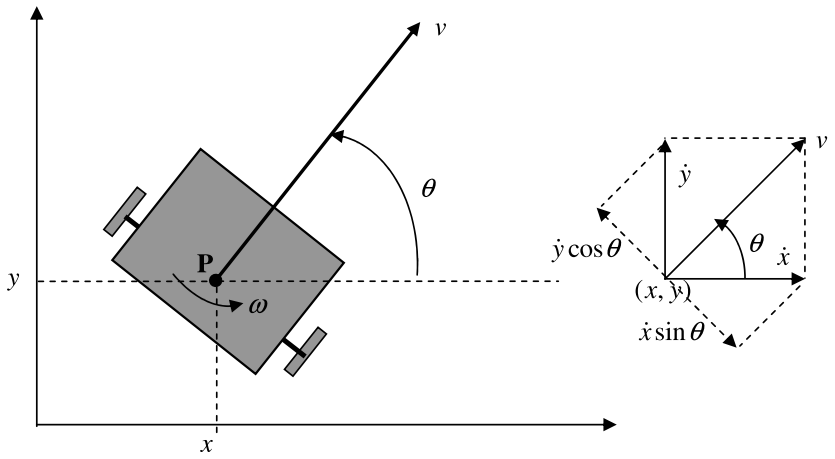

Figure 6.1: The unicycle ($\mathbb{R}^2 \times S^1$) (The two wheels collapse into one single wheel located at mid point)

So we build the corresponding accessibility distribution

$$\Delta_{\mathbf{G}'(\mathbf{q})} = \text{span}\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q}), \mathbf{g}_3(\mathbf{q}), \mathbf{g}_4(\mathbf{q})\} \tag{6.5}$$

We see that dim $\Delta_{\mathbf{G}'(\mathbf{q})} = n$ that means $\mathbf{G}'(\mathbf{q})$ has full rank. Hence, by Frobenius theorem the system is not integrable, i.e., the system is completely nonholonomic.

6.2 Unicycle

The kinematic model of the unicycle with the state $\mathbf{q} = (x, y, \theta)^T$ is described by the following equation (Figure 6.1) [3]:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu \\ \omega \end{bmatrix} \tag{6.6}$$

Where ν, ω are linear and angular velocities of the unicycle, and x, y is the position of the unicycle, while θ describes its orientation. Eliminating the input ν from Eq. (6.6) leads to the following kinematic rolling constraint which is not integrable ($k = k_{nh} = 1$) :

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0 \tag{6.7}$$

which implies no side-slip condition under the assumption of rolling contact. That means the instantaneous velocity of the point of contact of the unicycle is equal to zero at all times. So we get the following vector fields (see also Chapter 4.2.1)

$$\mathbf{g}_1(\mathbf{q}) = [\cos\theta \quad \sin\theta \quad 0]^T \tag{6.8}$$

$$\mathbf{g}_2(\mathbf{q}) = [0 \quad 0 \quad 1]^T \tag{6.9}$$

dim $\Delta_{\mathbf{G}(\mathbf{q})} = \text{span}\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q})\} = 2$, which is non-singular. Applying Lie bracket operation

$$\mathbf{g}_3(\mathbf{q}) = [\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q})] = [\sin\theta \quad -\cos\theta \quad 0]^T \tag{6.10}$$

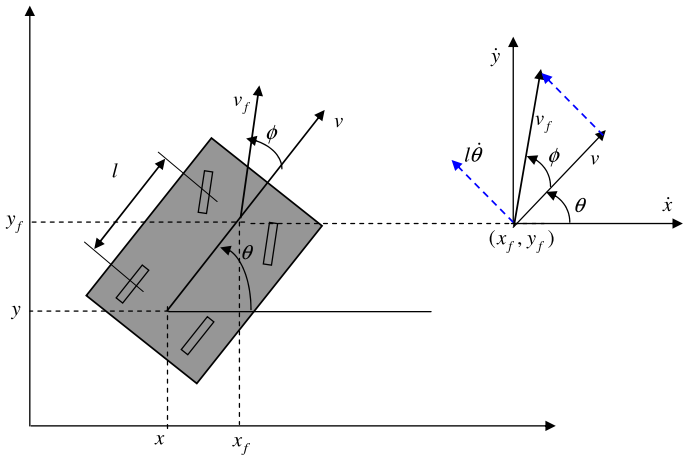

Figure 6.2: Car-like Robot $\mathbb{R}^2 \times S^1 \times \times S^1$ (The right sketch: $tan\phi = \frac{l\dot{\theta}}{\nu}$)

Then we get the following accessibility distribution

$$\mathbf{G}'(\mathbf{q}) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{bmatrix} \tag{6.11}$$

So the rank of $\mathbf{G}'(\mathbf{q}) = 3$ (full rank) and the determinant of $\mathbf{G}'(\mathbf{q})$ is non-zero for all $\mathbf{q} \in \mathbb{R}^3$, which means that $\mathbf{g}_3(\mathbf{q})$ is linearly independent of $\mathbf{g}_1(\mathbf{q})$ and $\mathbf{g}_2(\mathbf{q})$. Hence, by Frobenius theorem the system is not integrable, i.e. the system is completely nonholonomic.

6.3 Car-like Robot

Consider a mobile robot having the same kinematics as an automobile, as shown in Figure 6.2. For simplicity we assume that the two wheels on each axis collapse into a single wheel located at the mid point of the axis (bicycle model). The front wheel can be steered while the rear wheel orientation is fixed [1].

The generalized coordinates are $\mathbf{q} = (x, y, \theta, \phi)^T$, where (x, y) are Cartesian coordinates of rear wheel axle midpoint, ϕ is the steering angle with respect to car axis, and θ measures the orientation of car body with respect to the x-axis.

The system is subject to the following nonholonomic constraints on front-axle and rear-axle mid point respectively (see Figure 6.2):

$$\dot{x}_f \sin(\theta + \phi) - \dot{y}_f \sin(\theta + \phi) = 0 \tag{6.12}$$

$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0 \tag{6.13}$$

Further we have two holonomic constraints $k_h = 2$:

$$\dot{x}_f = x + l\cos\theta \tag{6.14}$$

$$\dot{y}_f = y + l\cos\theta \tag{6.15}$$

where l is the distance between the axles. And by time differentiation we obtain

$$\dot{x}_f = \dot{x} - l\dot{\theta}\sin\theta \tag{6.16}$$

$$\dot{y}_f = \dot{y} - l\dot{\theta}\cos\theta \tag{6.17}$$

Eliminating $\dot{x}_f$ and $\dot{y}_f$ into Eq. (6.12) and we get

$$\dot{x}\sin(\theta+\phi) - \dot{y}\sin(\theta+\phi) - l\dot{\theta}\cos\phi = 0 \tag{6.18}$$

Now we can write Eq. (6.13) and Eq. (6.18) in Pfaffian form

$$\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = 0 \tag{6.19}$$

$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -l\cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = 0 \tag{6.20}$$

Let us consider the following input vector

$$\mathbf{u} = (u_1 \quad u_2)^T = (\nu \quad \omega_f)^T \tag{6.21}$$

where ν is the linear velocity of the car wheels and $\omega_f = \dot{\phi}$ is the steering angular velocity. So we get the following kinematic model.

$$\dot{x} = \nu\cos\theta \tag{6.22}$$

$$\dot{y} = \nu\sin\theta \tag{6.23}$$

$$\dot{\theta} = \frac{\nu}{l}\tan\phi \tag{6.24}$$

$$\dot{\phi} = \omega_f \tag{6.25}$$

and finally in vector-matrix formulation:

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{1}{l}\tan\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu \\ \omega_f \end{bmatrix} \tag{6.26}$$

According to Eq. (3.18) we can express Eq. (6.26) in the following form

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q})\mathbf{u} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q})u_j, \qquad m = n - k \tag{6.27}$$

where $n = 4$ and $k = k_{nh} = 2$. $\mathbf{G}(\mathbf{q})$ is the nullspace of constraints matrix $\mathbf{A}(\mathbf{q})$ and $\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q})\}$ is a basis for this space, (see Appendix B). Finally we get the following kinematic model:

$$\dot{\mathbf{q}} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\phi \\ 0 \end{bmatrix} \nu + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega_f = \mathbf{g}_1(\mathbf{q})u_1 + \mathbf{g}_2(\mathbf{q})u_2 \tag{6.28}$$

The first vector field $\mathbf{g}_1$ tells us the direction we would go for a fixed ϕ and the second vector field $\mathbf{g}_2$ physically means that the steering depends only on ϕ. By using Lie Bracket operation (see Appendix C) we get

$$\mathbf{g}_3(\mathbf{q}) = [\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q})] = \begin{bmatrix} 0 & 0 & -\dfrac{1}{l\cos^2\phi} & 0 \end{bmatrix}^T \tag{6.29}$$

$$\mathbf{g}_4(\mathbf{q}) = [\mathbf{g}_1(\mathbf{q}), \mathbf{g}_3(\mathbf{q})] = \begin{bmatrix} -\dfrac{\sin\theta}{l\cos^2\phi} & \dfrac{\cos\theta}{l\cos^2\phi} & 0 & 0 \end{bmatrix}^T \tag{6.30}$$

So we build the corresponding accessibility distribution

$$\Delta_{\mathbf{G}'(\mathbf{q})} = \text{span}\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q}), \mathbf{g}_3(\mathbf{q}), \mathbf{g}_4(\mathbf{q})\} \tag{6.31}$$

We see that $\dim(\Delta_{\mathbf{G}'(\mathbf{q})}) = n$ that means $\mathbf{G}'(\mathbf{q})$ has full rank, so we conclude that according to the Frobenius Theorem the corresponding system is completely nonholonomic. But it should be carefully considered that singularity exists at $\phi = \pm\frac{\pi}{2}$ where the first vector field blows out.

Chapter 7

Mobile Redundant Robot

Inverse kinematics now is introduced as a means to create nonholonomic behavior. It should be noted that we have no kinematically nonholonomic constraints as in the examples before. Here we will consider simple examples for studying the kinematic behavior of manipulators with rotary joints that are based on moving base. But the dynamics of the system i.e., the conservation of linear and angular momentum is not being considered. This then will lead to the more general study for a free-floating planar space robot (Chapter 8) and finally to a 3D space robot (Chapter 9). In Chapter 8 and 9 full dynamics behavior is considered.

For simplicity we will consider the following planar mobile base with manipulator and its end effector (EE) which is performing a 2-dimensional task. Here the joint coordinate vector is $\mathbf{q} = (q_1, q_2, q_3, q_4)^T$ and the given EE task vector is $\xi = (\xi_1, \xi_2)^T$.

7.1 Inverse Kinematics

Here we will get the relations between the EE location and the mobile manipulator configuration. We know that Forward Kinematics (FK) expresses the EE location as a function of the mobile manipulator configuration. In our case the EE location is described by $\xi = (\xi_1, \xi_2)^T$ where as (ξ_1, ξ_2) are Cartesian coordinates of point O_{EE}. In our example FK writes:

$$\xi_1 = q_1 + l_1 \cos q_3 + l_2 \cos q_4 \tag{7.1}$$

$$\xi_2 = q_2 + l_1 \cos q_3 + l_2 \cos q_4 \tag{7.2}$$

We can write

$$\xi = \mathbf{f}(\mathbf{q}) \tag{7.3}$$

with $\xi = (\xi_1, \xi_2)^T$ and for the corresponding velocities

$$\dot{\xi} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \tag{7.4}$$

with Jacobian matrix

$$\mathbf{J}(\mathbf{q}) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 0 & -l_1 \cos q_3 & -l_2 \cos q_4 \\ 0 & 1 & -l_1 \cos q_3 & -l_2 \cos q_4 \end{bmatrix} \tag{7.5}$$

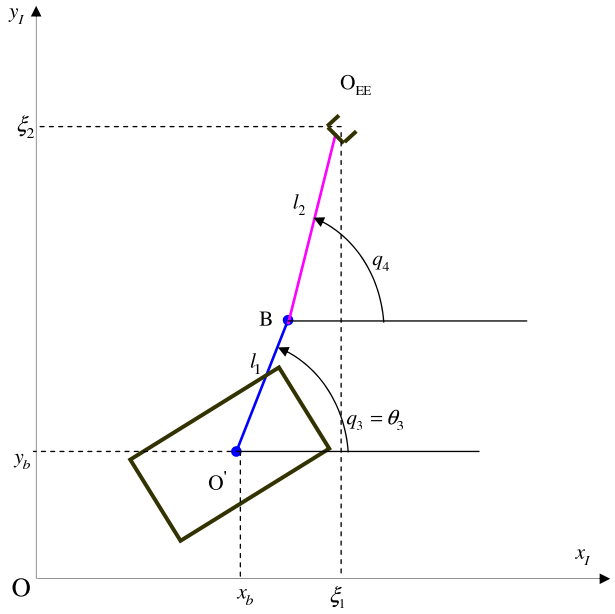

Figure 7.1: Mobile Robot

and $\mathbf{q} = (q_1, q_2, q_3, q_4)^T = (x_b, y_b, \theta_3, \theta_4)^T$.

The Jacobian matrix $\mathbf{J}(\mathbf{q})$ is not a square matrix. In order to calculate the velocities $\dot{\mathbf{q}}$, it is common to take the pseudoinverse:

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}(\mathbf{q})\dot{\xi} \tag{7.6}$$

where

$$\mathbf{J}^{\dagger} = \mathbf{J}^T(\mathbf{q})(\mathbf{J}(\mathbf{q})\mathbf{J}^T(\mathbf{q}))^{-1} \tag{7.7}$$

But in our case we will choose the weighted pseudoinverse $\mathbf{J}_w^{\dagger}(\mathbf{q})$,

$$\mathbf{J}_w^{\dagger}(\mathbf{q}) = \mathbf{W}^{-1}\mathbf{J}^T(\mathbf{q})(\mathbf{J}(\mathbf{q})\mathbf{J}^T(\mathbf{q}))^{-1} \tag{7.8}$$

where W is a weighted matrix that can be chosen appropriately in such a way that we have the possibility to tune the system to exhibit no additional singularities. In our particular case we will choose

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w^2 \end{bmatrix} \tag{7.9}$$

and hence

$$\mathbf{W}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{w^2} \end{bmatrix} \tag{7.10}$$

Table 7.1: The different cases for redundant robot

Case	Number of links	Geometry	DOF	EE Trajectory	Others
1	1	$l = l_1 + l_2 = 1.5\,\mathrm{m}$	3	Closed circular path	Using $\mathbf{J}_w^\dagger(\mathbf{q})$
2	2	$l_1 = 1.0\,\mathrm{m}$	4	Closed circular path	Using$\mathbf{J}_w^\dagger(\mathbf{q})$
		$l_2 = 1.5\,\mathrm{m}$			

where w is a non-dimensional numerical value. Regarding $\mathbf{J}_w^\dagger(\mathbf{q})$ into Eq. (7.6) leads to

$$\dot{\mathbf{q}} = \mathbf{J}_w^\dagger(\mathbf{q})\dot{\xi} \tag{7.11}$$

Although we did not start with description of nonholonomic constraints in Pfaffian form we can consider Eq. (7.11) as an equivalent kinematic model of the system where

$$\mathbf{J}_w^\dagger(\mathbf{q}) = \mathbf{G}(\mathbf{q}) \tag{7.12}$$

$$\dot{\xi} = \mathbf{u} \tag{7.13}$$

7.2 Case Studies

Now we will investigate two different cases which are shown in Table 7.1. Case 1 is a one link system and case 2 is a two link system.

7.2.1 Case 1: One Link Manipulator

The rotating joint at point B is blocked, i.e. $q_4 = 0$ for all time. We have only one link with $l = l_1 + l_2$ that is hinged in O'.
Now Eq. (7.1) and Eq. (7.2) are reduced to

$$\xi_1 = q_1 + l\cos q_3 \tag{7.14}$$

$$\xi_2 = q_2 + l\sin q_3 \tag{7.15}$$

and the Jacobian is

$$\mathbf{J}(\mathbf{q}) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 0 & -l\sin q_3 \\ 0 & 1 & l\cos q_3 \end{bmatrix} \tag{7.16}$$

The weighted pseudoinverse, $\mathbf{J}_w^\dagger(\mathbf{q})$ has been calculated by using Maple software (see Appendix G). This gives

$$\mathbf{J}_w^\dagger(\mathbf{q}) = \mathbf{G}(\mathbf{q}) = \frac{1}{w^2 + l^2} \begin{bmatrix} w^2 + l^2\cos^2 q_3 & l^2\cos q_3 \sin q_3 & \\ l^2\cos q_3 \sin q_3 & w^2 + l^2 - l^2\cos^2 q_3 & l^2 \\ -l\sin q_3 & l\cos q_3 & \end{bmatrix} \tag{7.17}$$

where

$$\mathbf{g}_1(\mathbf{q}) = \frac{1}{w^2 + l^2} \begin{bmatrix} w^2 + l^2 \cos^2 q_3 \\ l^2 \cos q_3 \sin q_3 \\ -l \sin q_3 \end{bmatrix} \tag{7.18}$$

$$\mathbf{g}_2(\mathbf{q}) = \frac{1}{w^2 + l^2} \begin{bmatrix} l^2 \cos q_3 \sin q_3 \\ w^2 + l^2 - l^2 \cos^2 q_3 \quad l^2 \\ l \cos q_3 \end{bmatrix} \tag{7.19}$$

After Lie bracket operation we obtain the following $\mathbf{g}_3(\mathbf{q})$ vector in which the system is possible to move in this direction:

$$\mathbf{g}_3(\mathbf{q}) = \frac{l^2}{(w^2 + l^2)^2} \begin{bmatrix} l \sin q_3 \\ -l \cos q_3 \\ l \end{bmatrix} \tag{7.20}$$

The accessibility distribution is

$$\Delta_{\mathbf{G}'(\mathbf{q})} = \text{span}\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q}), \mathbf{g}_3(\mathbf{q})\} \tag{7.21}$$

and in order to have the system being nonholonomic, we must show that

$$\dim\left(\Delta_{\mathbf{G}'(\mathbf{q})}\right) = 3 \tag{7.22}$$

$$\det\left(\Delta_{\mathbf{G}'(\mathbf{q})}\right) \neq 0 \tag{7.23}$$

But these two criteria are expected to depend on the value given for w . One necessary condition is

$$\dim\left(\Delta_{\mathbf{G}'(\mathbf{q})}\right) = \dim\left(\text{span}\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q}), \mathbf{g}_3(\mathbf{q})\}\right) = 2 \tag{7.24}$$

which is satisfied for any number $\mid w \mid > 0$, and $\mathbf{G}(\mathbf{q})$ becomes nonsingular. As we discussed before in Chapter 5, we can remark that the system is nonholonomic since $\mathbf{G}'(\mathbf{q})$ has full rank.

Special comments

If $w \to \infty$ then $g_1(\mathbf{q})$ and $g_2(\mathbf{q})$ become

$$\mathbf{g}_1(\mathbf{q}) = (1 \quad 0 \quad 0)^T \tag{7.25}$$

and

$$\mathbf{g}_2(\mathbf{q}) = (0 \quad 1 \quad 0)^T \tag{7.26}$$

Then a simple computation shows that $\mathbf{g}_3(\mathbf{q}) = (0 \quad 0 \quad 0)^T$ (Eq. 7.20), so that $\Delta_{\mathbf{G}(\mathbf{q})}$ is involutive and hence the system becomes holonomic.

In the simulation it has been shown that how system behaves on different values of w and sampling time t_s (Figure 7.2). Let the system base points O' reaches at (x_{bf}, y_{bf}) after one cycle i.e. $t = 2\pi$ sec. Then we get the distance between the initial and final position as follows

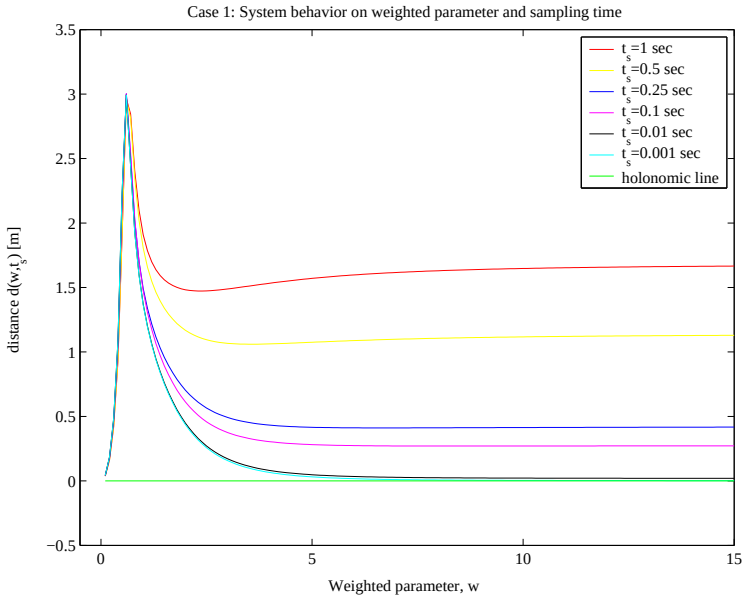

Figure 7.2: System behavior on weighted parameter

$$d(w, t_s) = \sqrt{(x_{bf} - x_{b0})^2 + (y_{bf} - y_{b0})^2} \tag{7.27}$$

Figure 7.2 illustrates that the system comes to touch the holonomic line sharply just after $w > 4$ as well as small sampling time has an important role on the system behavior.

Holonomic Line

Let us consider $\mathbf{g}_1(\mathbf{q}) = (1 \quad 0 \quad 0)^T$ and $\mathbf{g}_2(\mathbf{q}) = (0 \quad 1 \quad 0)^T$. Then we find that the system reaches its original position after one cycle. We get the nearly zero distance between the final and original position depending on sampling time ((Figure 7.3),(Figure 7.4)). So taking infinitesimal sampling time ε, the system finally reaches its original position just after one cycle. So for a fully holonomic system $d(w \to \infty, t_s \to \varepsilon)$ would not be varied on weighted parameter only but also depends on sampling time. But for a fully holonomic system $t_s \to \varepsilon$ that simply means distance, $d(w \to \infty, t_s \to \varepsilon)$, would be always zero after one cycle.

We already mentioned that $\mathbf{G}(\mathbf{q})$ being nonsingular is a necessary condition to investigate whether the system is holonomic or nonholonomic. Here this necessary condition is not violated because $\mathbf{g}_1(\mathbf{q})$ and $\mathbf{g}_2(\mathbf{q})$ are linearly independent vectors. So we can define for a fully holonomic system $d(w \to \infty, t_s \to \varepsilon) = 0$ which is considered as zero line or holonomic line (Figure 7.2).

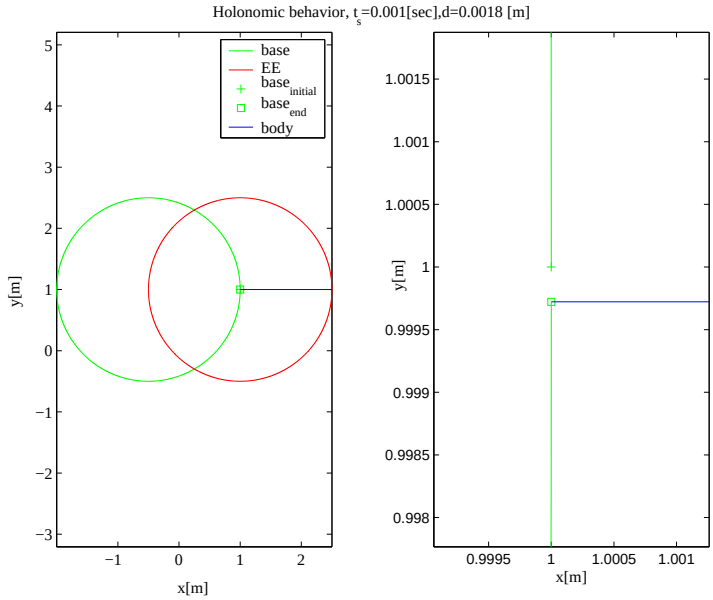

Figure 7.3: Holonomic behavior for $t_s = 0.001$sec (The left figure is a zoom of right figure)

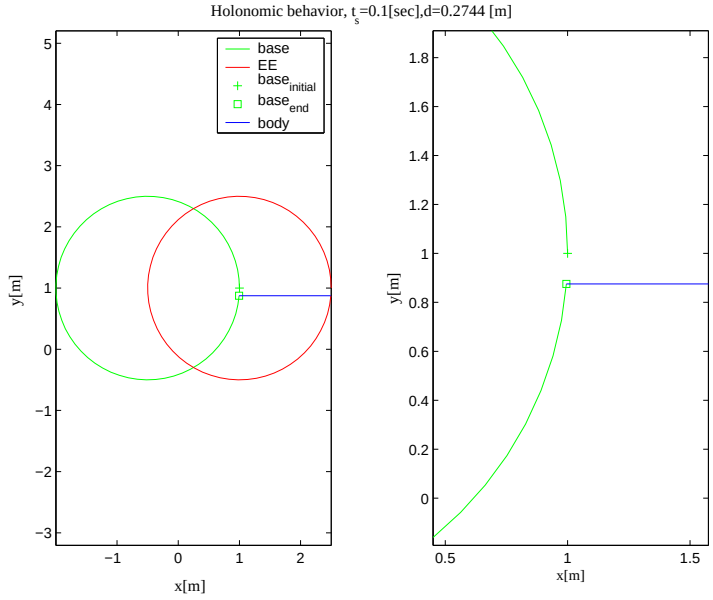

Figure 7.4: Holonomic behavior $t_s = 0.01$sec (The left figure is a zoom of right figure)

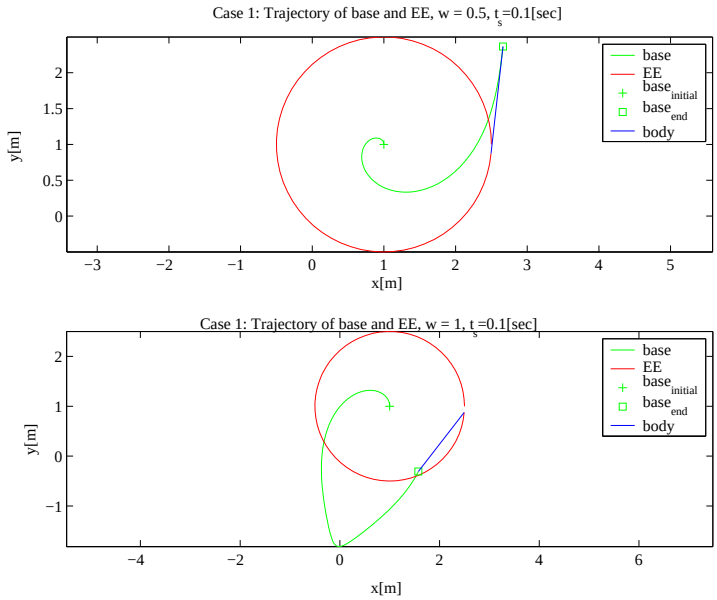

Figure 7.5: Holonomic behavior (Trajectory of base and the EE ($w = 0.5, 1$))

7.2.1.1 Simulation

At time $t = 0$ we assume $q_1 = x_{b0}$, $q_2 = y_{b0}$ and $q_3 = q_{30}$ with radius l and center of circle fixed at $(x_{b0}, y_{b0}) = (1, 1)$. Now we want to prescribe a circular motion for EE. This means

$$\xi_1 = x_{b0} + l\cos\theta_3 \tag{7.28}$$

$$\xi_2 = y_{b0} + l\sin\theta_3 \tag{7.29}$$

giving as control input

$$\mathbf{u} = \begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}-l\dot{\theta}_3\sin\theta_3\\ l\dot{\theta}_3\cos\theta_3\end{bmatrix} \tag{7.30}$$

with any time function for θ_3 or $\dot{\theta}_3$. We choose $\dot{\theta}_3 = \dot{q}_3 = \text{constant} = \omega$, $\theta_3 = \theta_{30} + \omega t$ and hence

$$\dot{\xi}_1 = -l\omega\sin(\theta_{30} + \omega t) \tag{7.31}$$

$$\dot{\xi}_2 = l\omega\cos(\theta_{30} + \omega t) \tag{7.32}$$

Let $\omega = 1$ rad/sec, then after one cycle i.e., $t = 2\pi$ sec the EE will reach its initial position. We take different weighted parameters while taking the same sampling time and found that increasing the weighted parameter the base is approaching its initial position. Using the high value of weighted parameter this nonholonomic system becomes holonomic in nature (Figure 7.5, Figure 7.6).

7.2.2 Case 2: Two Link Manipulator

In this case we consider two links (Figure 7.1). Then we will get the following Jacobian Matrix .

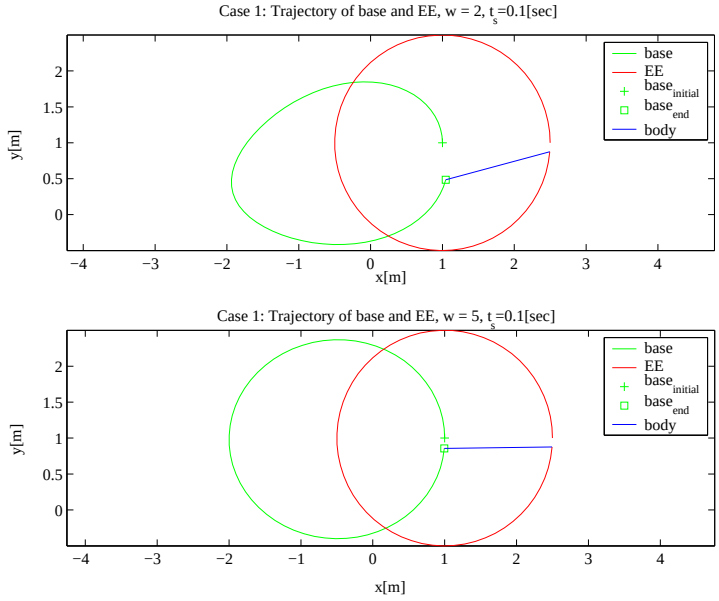

Figure 7.6: Holonomic behavior (Trajectory of base and the EE ($w = 2, 5$))

$$\mathbf{J}(\mathbf{q}) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 0 & -l_1 \sin q_3 & -l_2 \sin q_4 \\ 0 & 1 & l_1 \cos q_3 & l_2 \cos q_4 \end{bmatrix} \tag{7.33}$$

Taking weighted pseudoinverse, $\mathbf{J}_w^{\dagger}(\mathbf{q})$ we get the following $\mathbf{g}_1(\mathbf{q})$ and $\mathbf{g}_2(\mathbf{q})$ vectors (see Appendix G):

$$\mathbf{g}_1(\mathbf{q}) = \alpha \begin{bmatrix} -w^2 - l_1^2 w^2 \cos^2 q_3 - l_2^2 \cos^2 q_4 \\ -l_1^2 w^2 \cos q_3 \sin q_3 - l_2^2 w^2 \cos q_4 \sin q_4 \\ l_1 w^2 \sin q_3 + l_2^2 \sin q_3 \cos^2 q_4 - l_2^2 \cos q_3 \sin q_4 \cos q_4 \\ l_2(-\sin q_4 - l_1^2 \sin q_4 \cos^2 q_3 + l_1^2 \cos q_3 \cos q_4) \end{bmatrix} \tag{7.34}$$

$$\mathbf{g}_2(\mathbf{q}) = \alpha \begin{bmatrix} l_1 w^2 \sin q_3 + l_2^2 \sin q_3 \cos^2 q_4 - l_2^2 \cos q_3 \sin q_4 \cos q_4 \\ -w^2 - l_1^2 w^2 + l_1^2 w^2 \cos^2 q_3 - l_2^2 + l_2^2 \sin q_4 \cos q_4 \cos^2 q_4 \\ l_1 l_2 w^2 \sin q_3 \sin q_4 \cos q_4 - w^2 \cos^2 q_3 + l_2^2 \cos q_3 \cos^2 q_4 \\ l_2(l_1^2 \sin q_3 \sin q_4 \cos q_3 - \cos q_4 - l_1^2 \cos^2 q_3 \cos q_4) \end{bmatrix} \tag{7.35}$$

where

$\alpha = (-w^2 - l_1^2 w^2 - l_1^2 l_2^2 \cos^2 q_4 + 2 l_1^2 l_2^2 \cos^2 q_4 \cos^2 q_3 - l_2^2 - l_1^2 l_2^2 \cos^2 q_3 + 2 l_1^2 l_2^2 \sin q_3 \cos q_3 \sin q_4 \cos q_4)^{-1}$

As $\dim(\Delta_{\mathbf{G}(\mathbf{q})}) = \dim(\text{span}\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q})\}) = 2$, we can further investigate for its holonomic or nonholonomic behavior. Using Maple software we get $\mathbf{g}_3(\mathbf{q})$ and $\mathbf{g}_4(\mathbf{q})$ (see Appendix G). The dimension of the accessibility distribution

$$\dim(\Delta_{\mathbf{G}'(\mathbf{q})}) = \dim(\text{span}\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q}), \mathbf{g}_3(\mathbf{q}), \mathbf{g}_4(\mathbf{q})\}) = 4 \tag{7.36}$$

So we can easily conclude that the system is nonholonomic as well (Chapter 5).

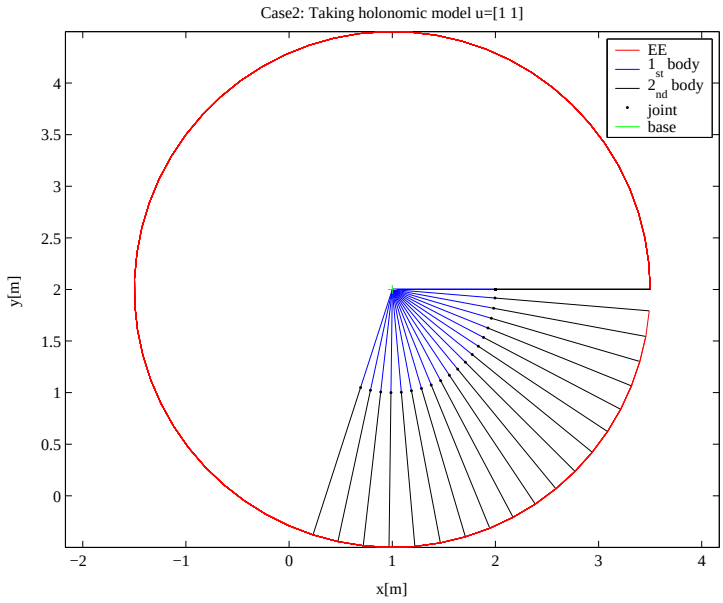

Figure 7.7: Holonomic model (Two link)

7.2.2.1 Simulation

To investigate the system behavior of two link manipulator. First we consider the holonomic model in such a way that

$$\mathbf{g}_1(\mathbf{q}) = (0 \quad 0 \quad 1 \quad 0)^T \tag{7.37}$$

and

$$\mathbf{g}_2(\mathbf{q}) = (0 \quad 0 \quad 0 \quad 1)^T \tag{7.38}$$

Then a simple computation shows that

$$\mathbf{g}_3(\mathbf{q}) = (0 \quad 0 \quad 0 \quad 0)^T \tag{7.39}$$

with $\mathbf{u} = (\omega_1 \quad \omega_2)^T$ where $\omega_1 = \omega_2 = 1$ rad/sec.

Then we find for a circular end effector trajectory that the base returns to its original position at (x_{b0}, y_{b0}) after one cycle (Figure 7.7).

Now we consider our computed $\mathbf{g}_1(\mathbf{q})$ and $\mathbf{g}_2(\mathbf{q})$ vectors with same input $\mathbf{u} = (1 \quad 1)^T$ and we find in the following Figure 7.8 how the system behaves after one cycle. Figure 7.8 illustrates that the base has no tendency to reach its original position. Moreover, we don't get the circular EE trajectory as expected (Figure 7.7). That means the system itself is nonholonomic in nature.

In the next Figure 7.9 it is shown how the system behaves on weighted parameter variations. As we discussed in previous section here we found that the distance between the final and initial position at beginning is much higher than previous system but after few steps its nearly constant over weighted parameter w but slightly varied on sampling time t_s compared to previous the system.

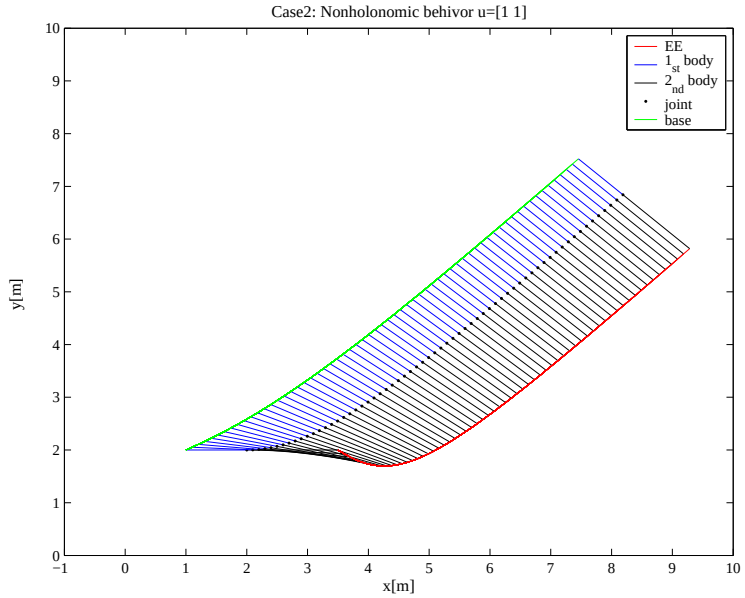

Figure 7.8: Nonholonomic behavior (Two link)

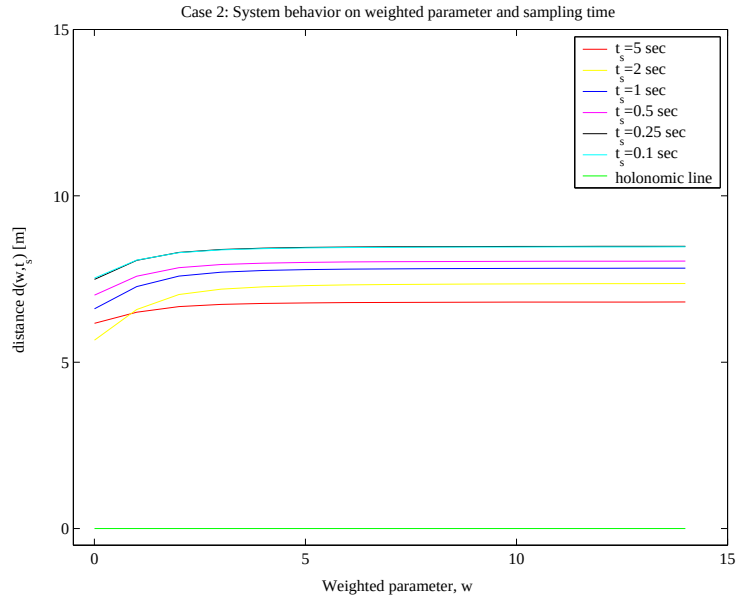

Figure 7.9: System behavior on weighted parameter (Case 2)

7.3 Comments

It has already been shown that though the EE performing a cyclic task i.e. it traces a closed path in the workspace, the base of a kinematically redundant manipulator does not return to its original position. But the final position of base depends on weighted parameter and sampling time that means the trajectory of the base depends on the choices of $\mathbf{J}^{\dagger}(\mathbf{q})$.

Finally we get the following:
In case 1, the system can switch from nonholonomic to holonomic if and only if $w \to \infty$. But in case 2, we get circular EE trajectory in holonomic model (Figure 7.7) but we don't get the circular EE trajectory in two link manipulator redundant system i.e., the system never goes to holonomic. So we can conclude that the redundancy is one of the sources of producing nonholonomic constraints.

Chapter 8

Free-Floating Space Robot: Plane Motion

Here dynamics of the system is considered, which will result in nonholonomic constraints. Additionally with the given EE trajectory and IK we would end up with increased nonholonomy compared to the example of Chapter 7, where dynamics has been neglected.

However, in order not to complicate the problem to be studied here, we neglect to specify the EE trajectory. The nonholonomic nature then is only given by conservation of angular momentum.

In this chapter we will discuss about the multibody space robots with planar structure. Usually the base attitude of the space robot can be normally controlled by reaction wheels or thrust jets. When thrust jets are used to control the attitude of the base, the linear or angular momentum is not conserved, due to external thrust forces. In our case we consider that the robot will completely be free-floating in zero-gravity environment. When multibody systems are allowed to float freely that means without having a fixed base, both the linear and angular momentum are conserved. Due to the conservation of angular momentum, the non-integrable velocity constraints result in the nonholonomic nature of a space robot.

8.1 General Description

Consider an n-body planar open kinematic chain which floats freely as shown in Figure 8.1. The first body represents the base and the other $n-1$ bodies are the manipulator links. For the i-th body, let l_i be the hinge to hinge length and d_i the distance from joint $i-1$ to its center of mass. Further, denote by r_i and v_i, respectively, the position and linear velocity of the center of mass, and by ω_i the angular velocity expressed in an inertial frame. And m_i indicates the mass of the i-th body and J_i its inertia matrix with respect to the center of mass. When no external force is applied, and in the absence of gravity and dissipation forces, the linear and angular momentum of the multibody system is conserved. Assume that initially they are all zero. The law of conservation of linear momentum is written in general,

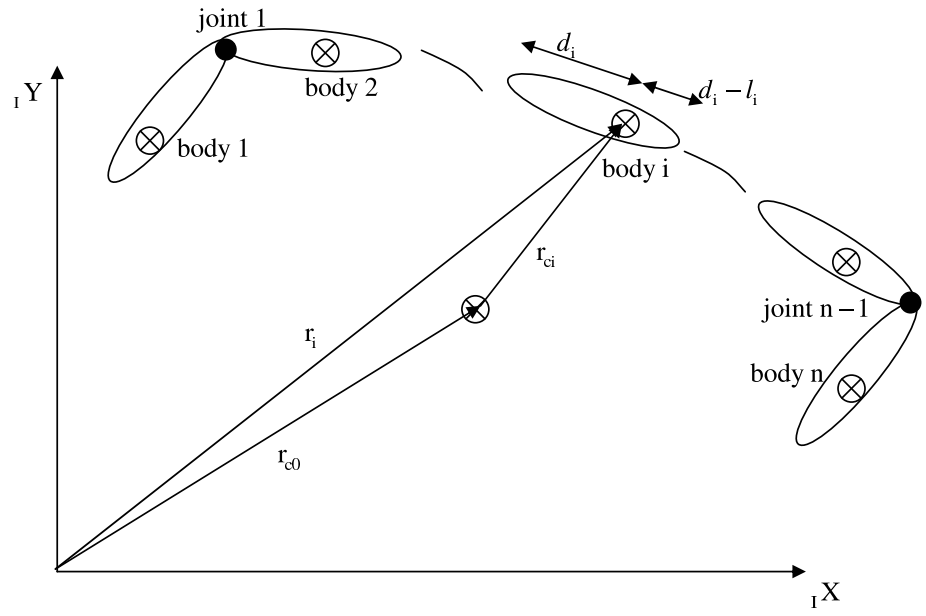

Figure 8.1: An body planar space structure: satellite (first body) manipulator

as

$$\sum_{i}^{n} m_i \mathbf{v}_i = \sum_{i}^{n} m_i \dot{\mathbf{r}}_i = 0 \tag{8.1}$$

That can be obviously integrated to

$$\sum_{i}^{n} m_i \mathbf{r}_i = m_t \mathbf{r}_{c0} = c \tag{8.2}$$

where m_t is the total mass of the system, $\mathbf{r}_{c0}$ position vector of the system center of mass and c is a vector constant. As a consequence , the conservation of linear momentum gives rise to three holonomic constraints, indicating that the system center of mass does not move. The conservation of angular momentum is expressed as

$$\sum_{i}^{n} \left[\mathbf{J}_i \omega_i + m_i \left(\mathbf{r}_i \times \mathbf{v}_i \right) \right] = 0 \tag{8.3}$$

These three differential constraints are, in general, nonholonomic in the three dimensional case. Since in the present case, motion is constrained to the xy-plane, Eq. (8.3) reduces to only one differential constraint. To perform the nonholonomic analysis, it will be first necessary to convert this constraint into a Pfaffian form of the system generalized coordinates $\mathbf{q}$.
There are two approaches to get the Pfaffian form from the conservation of linear and angular momentum.

1. First approach (Kinetic Energy)
2. Second approach(Direct approach)

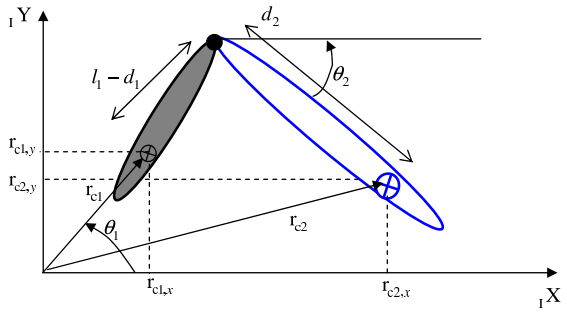

Figure 8.2: Two-body planar space robot

The first one is shown by De Luca [1], deriving the generalized momenta from the kinetic energy of an n-body system as follows

$$\mathbf{T} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{B}(\mathbf{q})\,\dot{\mathbf{q}} \tag{8.4}$$

where $\mathbf{B}(\mathbf{q})$ is the symmetric and positive definite inertia matrix and the generalized momenta are defined as

$$\mathbf{p}_i = \frac{\partial(\mathbf{T} - \mathbf{U})}{\partial \dot{\mathbf{q}}_1} = \mathbf{b}_i^T(\mathbf{q})\,\dot{\mathbf{q}},\ i = 1, \ldots, n \tag{8.5}$$

where $\mathbf{b}_i$ is the i-th column of $\mathbf{B}$. Here we consider the potential energy, $\mathbf{U}$ is constant and it can be written into Pfaffian constrain as follows

$$\sum_i^n \mathbf{p}_i = \sum_i^n \mathbf{b}_i^T(\mathbf{q})\,\dot{\mathbf{q}} = \mathbf{1}^T \mathbf{B}(\mathbf{q})\,\dot{\mathbf{q}} \tag{8.6}$$

where $\mathbf{1}^T = (1, 1, \ldots, 1)$.
A second one is a more direct approach, which will be followed in the succeeding chapters.

8.2 Two-Body Planar Space Robot

8.2.1 Dynamic Model

Consider the following structure given in Figure 8.2. We choose the inertia frame at system center of mass, i.e., $\mathbf{r}_{c0}$.
We get the following geometrical constraints

$$\begin{bmatrix} r_{c1,x} \\ r_{c1,y} \end{bmatrix} + (l_1 - d_1)\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + d_2 + \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} = \begin{bmatrix} r_{c2,x} \\ r_{c2,y} \end{bmatrix} \tag{8.7}$$

From conservation of linear momentum we get

$$m_1 \begin{bmatrix} r_{c1,x} \\ r_{c1,y} \end{bmatrix} + m_2 \begin{bmatrix} r_{c2,x} \\ r_{c2,y} \end{bmatrix} = m_t \begin{bmatrix} r_{c0,x} \\ r_{c0,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8.8}$$

Multiply Eq. (8.7) by m_2 and from Eq. (8.8) we obtain

$$\begin{bmatrix} r_{c1,x} \\ r_{c1,y} \end{bmatrix} = k_{11} \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + k_{12} \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} \tag{8.9}$$

where $k_{11} = \frac{-m_2(l_1-d_1)}{m_t}$ and $k_{12} = \frac{-m_2 d_2}{m_t}$
Multiply Eq. (8.7) by m_1 and from Eq. (8.8) we obtain

$$\begin{bmatrix} r_{c2,x} \\ r_{c2,y} \end{bmatrix} = k_{21} \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + k_{22} \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} \tag{8.10}$$

where $k_{21} = \frac{m_1(l_1-d_1)}{m_t}$ and $k_{22} = \frac{m_1 d_2}{m_t}$.
Then we get

$$\mathbf{r}_{c1} = \begin{bmatrix} k_{11}\cos\theta_1 + k_{12}\cos\theta_2 \\ k_{11}\sin\theta_1 + k_{12}\sin\theta_2 \\ 0 \end{bmatrix}$$

and

$$\mathbf{r}_{c2} = \begin{bmatrix} k_{21}\cos\theta_1 + k_{22}\cos\theta_2 \\ k_{21}\sin\theta_1 + k_{22}\sin\theta_2 \\ 0 \end{bmatrix}$$

Now we will calculate the total angular momentum of the system by two approaches. We get the same equation from these two approaches (see Appendix E).

1. First Approach (Kinetic Energy)
   The calculation of the first approach is done in Appendix E.

2. Second Approach (Direct Approach)
   Here we will calculate the angular momentum of the 1^{st} and 2^{nd} body by conventional way.

Angular momentum of the 1^{st} body

$$\begin{aligned} \mathbf{L}_1 &= \mathbf{J}_1\omega_1 + m_1(\mathbf{r}_{c1} \times \mathbf{v}_1) \\ &= \mathbf{J}_1\dot{\theta}_1 + (\tilde{\mathbf{r}}_{c1} \cdot \mathbf{v}_1) \end{aligned} \tag{8.11}$$

where

$$\begin{aligned} \tilde{\mathbf{r}}_{c1} &= \begin{bmatrix} 0 & -r_{c1,z} & r_{c1,y} \\ r_{c1,z} & 0 & -r_{c1,x} \\ -r_{c1,y} & r_{c1,x} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & r_{c1,y} \\ 0 & 0 & -r_{c1,x} \\ -r_{c1,y} & r_{c1,x} & 0 \end{bmatrix} \end{aligned}$$

where

$$r_{c1,x} = k_{11}\cos\theta_1 + k_{12}\cos\theta_2$$
$$r_{c1,y} = k_{11}\sin\theta_1 + k_{12}\sin\theta_2$$

and

$$\mathbf{v}_1 = \begin{bmatrix} -k_{11}\dot{\theta}_1\sin\theta_1 - k_{12}\dot{\theta}_2\sin\theta_2 \\ -k_{11}\dot{\theta}_1\cos\theta_1 - k_{12}\dot{\theta}_2\cos\theta_2 \\ 0 \end{bmatrix}$$

So we obtained the linear momentum of the 1^{st} and 2^{nd} body respectively:

$$\begin{aligned} \mathbf{L}_1 &= \mathbf{J}_1\dot{\theta}_1 + m_1\left(k_{11}^2\dot{\theta}_1 + k_{12}^2\dot{\theta}_2\right) + m_1k_{11}k_{12}\left(\dot{\theta}_2\cos\left(\theta_2-\theta_1\right)\right) \\ &+ m_1k_{11}k_{12}\left(\dot{\theta}_1\cos\left(\theta_2-\theta_1\right)\right) \end{aligned} \tag{8.12}$$

Similarly

$$\begin{aligned} \mathbf{L}_2 &= \mathbf{J}_2\dot{\theta}_2 + m_2\left(k_{21}^2\dot{\theta}_1 + k_{22}^2\dot{\theta}_2\right) + m_2k_{21}k_{22}\left(\dot{\theta}_2\cos\left(\theta_2-\theta_1\right)\right) \\ &+ m_2k_{21}k_{22}\left(\dot{\theta}_1\cos\left(\theta_2-\theta_1\right)\right) \end{aligned} \tag{8.13}$$

So conservation of angular momentum (see Appendix E)

$$\mathbf{L}_1 + \mathbf{L}_2 = 0 \tag{8.14}$$

can be rewritten as

$$\begin{bmatrix} \overline{J}_1 + \overline{J}_2 + 2b_{12}(\phi_1) & \overline{J}_2 + b_{12}(\phi_1) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \end{bmatrix} = 0 \tag{8.15}$$

where

$$\begin{aligned} \phi_1 &= \theta_2 - \theta_1 \\ \overline{J}_1 &= J_1 + m_1k_{11}^2 + m_2k_{21}^2 \\ \overline{J}_2 &= J_2 + m_1k_{12}^2 + m_2k_{22}^2 \quad \text{and} \\ b_{12}(\phi_1) &= (m_1k_{11}k_{12} + m_2k_{21}k_{22})\cos(\phi_1) \end{aligned}$$

Let $\mathbf{A}(\mathbf{q}) = \begin{bmatrix} \overline{J}_1 + \overline{J}_2 + 2b_{12}(\phi_1) & \overline{J}_2 + b_{12}(\phi_1) \end{bmatrix}$ and we will find out the null space of $\mathbf{A}(\mathbf{q})$. The equation $\mathbf{A}(\dot{\mathbf{q}}) = 0$ ends up being just the single equation

$$\left(\overline{J}_1 + \overline{J}_2 + 2b_{12}(\phi_1)\right)\dot{\theta}_1 + \left(\overline{J}_2 + b_{12}(\phi_1)\right)\dot{\phi}_1 = 0 \tag{8.16}$$

We let $\dot{\phi}_1 = u, a = \frac{\overline{J}_2}{m_1k_{11}k_{12}+m_2k_{21}k_{22}}\overline{J}_2$, and $b = \frac{\overline{J}_1+\overline{J}_2}{m_1k_{11}k_{12}+m_2k_{21}k_{22}}$ which gives

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \end{bmatrix} = \begin{bmatrix} -\frac{a+\cos\phi_1}{b+2\cos\phi_1} \\ 1 \end{bmatrix} u = \mathbf{G}(\mathbf{q})u \tag{8.17}$$

8.2.2 Holonomic Characteristic of 2-Body Planar Robot

Now we will investigate the holonomic charactesistcs of 2-Body planar robot by the following two approaches.

1. Lie Bracket approach

From Eq. (8.17) we get the following distribution

$$\Delta_{\mathbf{G(q)}} = \{\mathbf{g}_1(\mathbf{q})\}$$

where

$$\mathbf{g}_1(\mathbf{q}) = \begin{bmatrix} -\frac{a+\cos\phi_1}{b+2\cos\phi_1} \\ 1 \end{bmatrix}$$

We know that the distribution generated by single vector field is always involutive. So it is not possible to generate other $\mathbf{g}$'s vectors. Hence we get the dimension of $\Delta_{\mathbf{G(q)}}$ is equal to m.

$$\dim \Delta_{\mathbf{G(q)}} = n - k = 2 - 1 = 1$$

Hence the system of 2-body planar space robot is holonomic (see Chapter 5).

2. Conventional approach

Conservation of angular momentum gives the following integrable differential constraints:

$$(b + 2\cos\phi_1)\,\dot{\theta}_1 + (a + \cos\phi_1)\,\dot{\phi}_1 = 0$$

from where we obtain immediately

$$\dot{\theta}_1 = -\frac{a + \cos\phi_1}{b + 2\cos\phi_1}\dot{\phi}_1 \tag{8.18}$$

or by regarding $\dot{\phi}_1 = \dot{\theta}_2 - \dot{\theta}_1$:

$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{a - b - \cos\phi_1}{a + \cos\phi_1} \tag{8.19}$$

In Figure 8.3 we have plotted the relative angle verses the ratio of $\dot{\theta}_2$ and $\dot{\theta}_1$ with m_1=40 kg, m_2=50 kg, $J_1 = J_2$=100 kg-m^2, l_1=5 m, l_2=20 m, d_1=2 m, d_2=5 m, i.e., a=1.997 and b=2.8667. And in 2nd case we take m_1=20 kg, m_2=25 kg, $J_1 = J_2$=100 kg-m^2, l_1=2.5 m, l_2=10 m, d_1=1 m, d_2=2.5 m, i.e., a=4.0667 and b=7.067.

We offer now an analytical solution for θ_1 depending from ϕ_1. From Eq. (8.18) we get

$$\dot{\theta}_1 = -\frac{a}{b + 2\cos\phi_1}\dot{\phi}_1 - \frac{\cos\phi_1}{b + 2\cos\phi_1}\dot{\phi}_1 \tag{8.20}$$

By taking the integral we obtain

$$\begin{aligned} \theta_1 &= -\frac{2a}{\sqrt{b^2-4}}\arctan\frac{(b-2)\tan\frac{\phi_1}{2}}{\sqrt{b^2-4}} \\ &- \frac{\phi_1}{2} + \frac{b}{\sqrt{b^2-4}}\arctan\frac{(b-2)\tan\frac{\phi_1}{2}}{\sqrt{b^2-4}} + c \end{aligned} \tag{8.21}$$

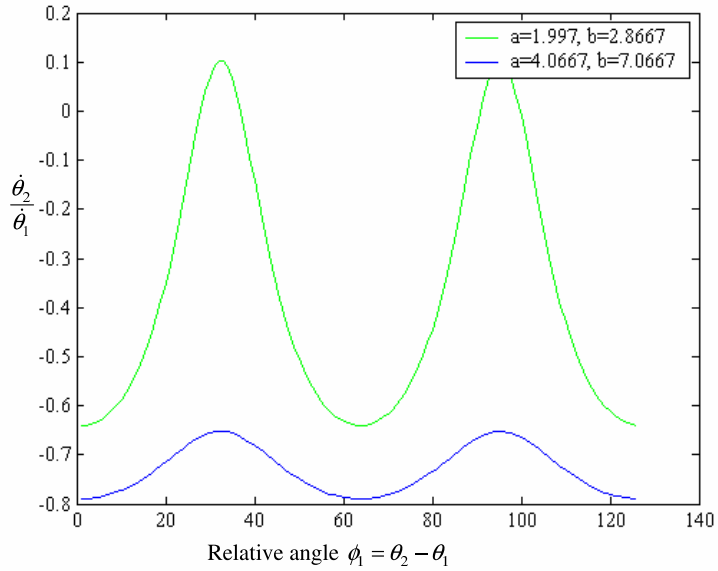

Figure 8.3: Holonomic characteristic in two-body planar robot

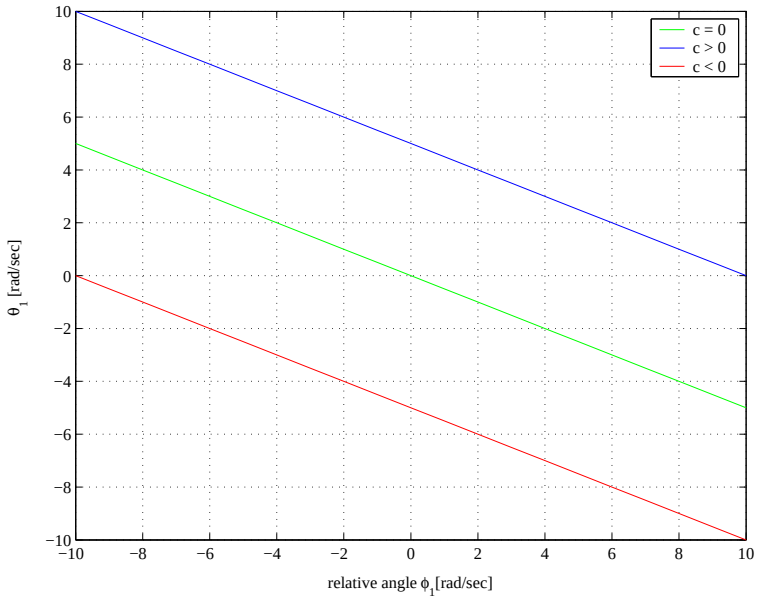

Figure 8.4: Relative angle, ϕ_1 verses θ_1 (different values of c)

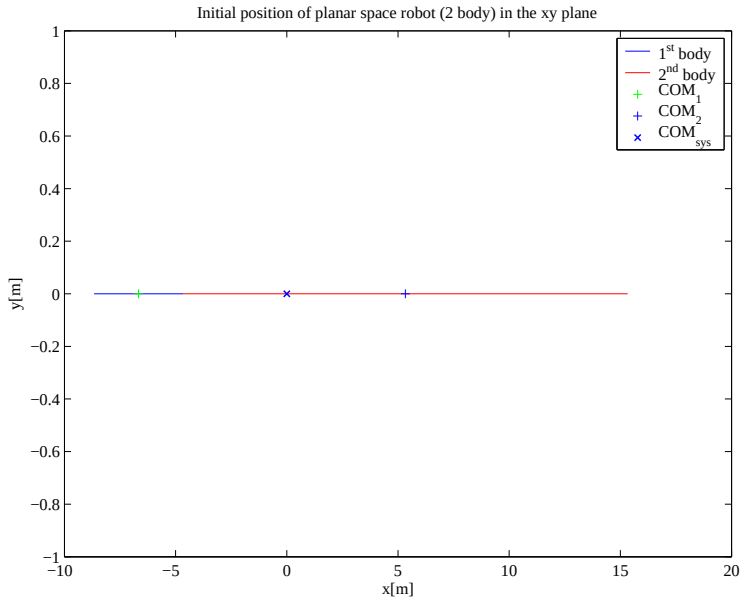

Figure 8.5: Planar space robot (2-body) at initial position

where c is a constant depending on the initial conditions.

If $\overline{J}_1 = \overline{J}_2$, Eq. (8.20) simplifies

$$\dot{\theta}_1 = -\frac{1}{2}\dot{\phi}_1 \tag{8.22}$$

by integration we easily obtain

$$\theta_1 = -\frac{1}{2}\phi_1 + c \tag{8.23}$$

Figure 8.4 shows the plot of Eq. (8.23) for different values of c.

8.2.3 Simulation Results

We will simulate the kinematic model of the two body planar space robot using $u = \omega_1$=1 rad/sec, m_1=40 kg, m_2=50 kg, l_1=4 m, l_2=20 m, d_1=2 m, d_2=10 m, $J_1 = J_2$=100 kg-m^2. Figure 8.5 shows the initial orientation of planar space robot in the xy plane i.e. $q_0 = (\ \theta_1 \quad \phi_1\ )_0 = (\ 0 \quad 0\ )$.

Figure 8.6 shows the variation of system center of mass over time and we find that the value of its component always zero while the value of its y-component varies little over time on the scale of 10^{-14}. Figure 8.7 shows that after $t = 7\pi$ sec and $t = 14\pi$ sec (approximately) the internal configurations of the system reach its initial values i.e., $q_0 = (\ \theta_1 \quad \phi_1\ )_0 = (\ 0 \quad 0\ )$ and we find that after $t = 14\pi$ sec the system itself return to its original posture (Figure 8.8).

Hence we can conclude that the two body planar space robot is a holonomic system. And we can not steer this system to any pair of (θ_1, ϕ_1) by using single input, u. But the trajectory of the system depends on the geometric properties of the bodies i.e., the mass, inertia and geometric length and position of center of mass as well. In the next section 8.3 we will discuss

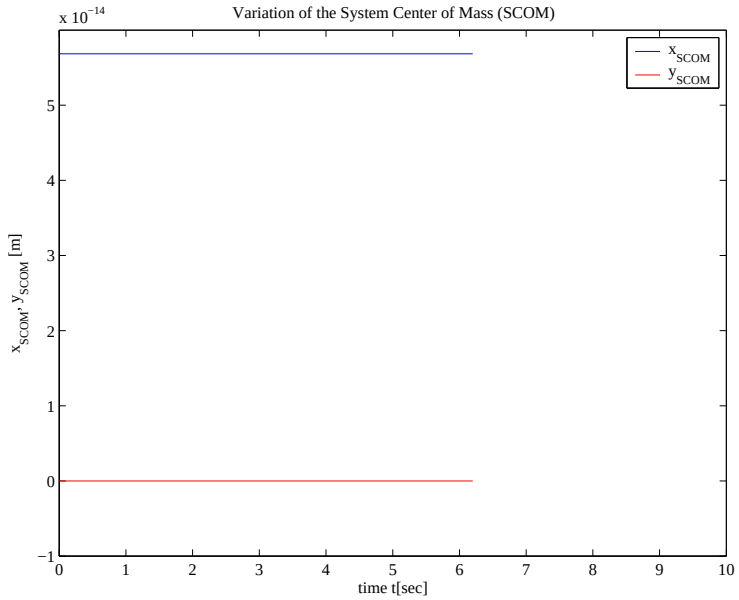

Figure 8.6: Variation of the system center of mass over time (2-body)

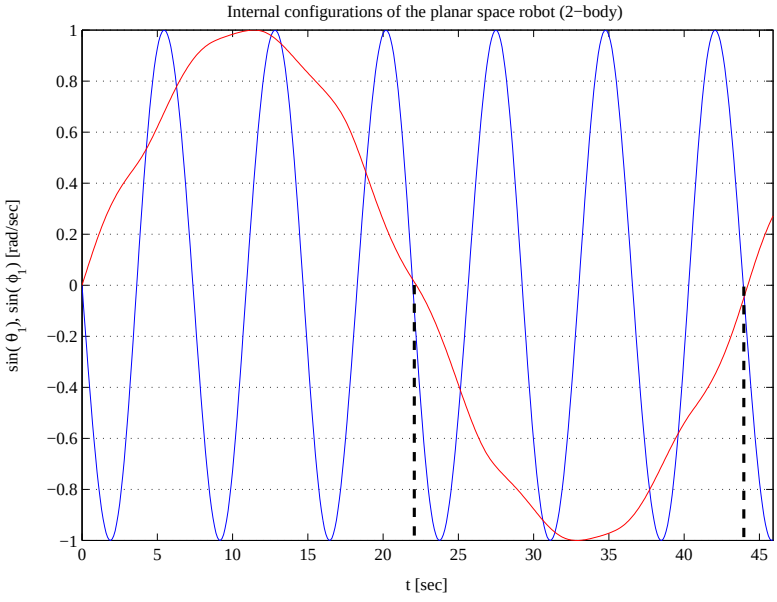

Figure 8.7: Internal configurations of the planar space robot (2-body)

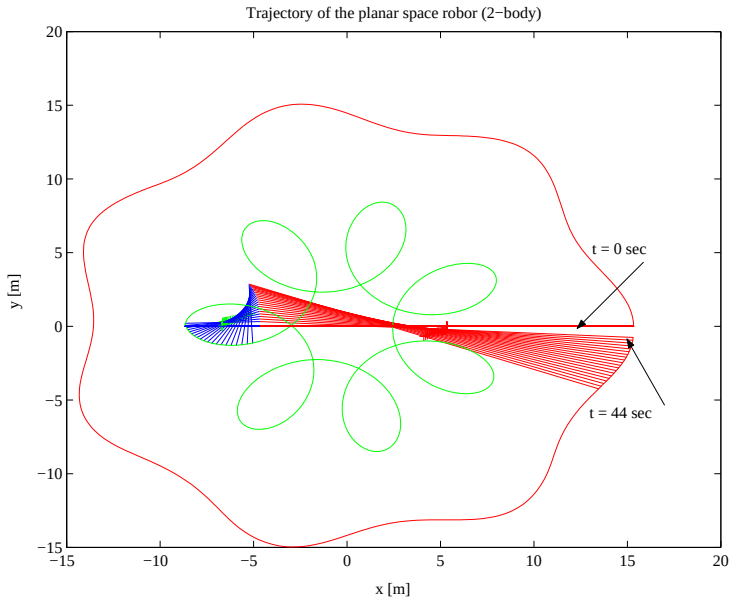

Figure 8.8: The (holonomic) trajectory of the planar space robot (2-body)

the three body planar space robot by considering the different cases for different geometric properties (Table 8.1).

8.3 Three-Body Planar Space Robot

8.3.1 Dynamic Model

Now we will calculate the total angular momentum of the system by two approaches. Here we also get the same equation for both approaches that is shown in Appendix E.

1. First approach (Kinetic Energy)
   The calculation of the first approach is done in Appendix E.

2. Second approach (Direct approach)
   The calculation of the second approach is shown as follows:

In the case of the three body planar space robot we get the following angular momentum of the 1^{st}, 2^{nd} and 3^{rd} body respectively.

$$\mathbf{L}_1 = \mathbf{J}_1\omega_1 + m_1(\mathbf{r}_{c1} \times \mathbf{v}_1) \tag{8.24}$$
$$\mathbf{L}_2 = \mathbf{J}_2\omega_2 + m_2(\mathbf{r}_{c2} \times \mathbf{v}_2) \tag{8.25}$$
$$\mathbf{L}_3 = \mathbf{J}_3\omega_3 + m_3(\mathbf{r}_{c3} \times \mathbf{v}_3) \tag{8.26}$$

where $\mathbf{J}_1$,$\mathbf{J}_2$,$\mathbf{J}_3$ are the rotational angular momentums, ω_1,ω_2,ω_3 are the angular velocities, $\mathbf{v}_1$,$\mathbf{v}_2$,$\mathbf{v}_3$ are the linear velocity at center of mass and $\mathbf{r}_{c1}$,$\mathbf{r}_{c2}$,$\mathbf{r}_{c3}$ are the position vector of center of mass of the 1^{st}, 2^{nd} and 3^{rd} body respectively.

As we described before the conservation of linear momentum gives holonomic constraints and can be written as

$$m_1\mathbf{r}_{c1} + m_2\mathbf{r}_{c2} + m_3\mathbf{r}_{c3} = m_t\mathbf{r}_{c0} = c \tag{8.27}$$

In this case we define that inertia frame is located at the origin. So Eq. (8.27) becomes

$$m_1\mathbf{r}_{c1} + m_2\mathbf{r}_{c2} + m_3\mathbf{r}_{c3} = m_t\mathbf{r}_{c0} = 0 \tag{8.28}$$

By adding the above three equations Eq. (8.24), Eq. (8.25), Eq. (8.26) we get the total angular momentum of the system. So we consider the conservation of angular momentum of the system as follows

$$\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 = 0 \tag{8.29}$$

Our primary concern is to get the kinematic model of the system. So we convert the constraint Eq. (8.29) into the following Pfaffian form (for calculation see Appendix E):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = 0 \tag{8.30}$$

where

$$\begin{aligned} a_{11} =& b_{11} + c_1\cos(\theta_1 - \theta_2) + c_2\cos(\theta_1 - \theta_3) \\ a_{12} =& c_1\cos(\theta_1 - \theta_2) + b_{22} + c_3\cos(\theta_2 - \theta_3) \\ a_{13} =& c_2\cos(\theta_1 - \theta_3) + c_3\cos(\theta_2 - \theta_3) + b_{33} \end{aligned}$$

and c_1,c_2,c_3 depend on only inertia and geometrical properties. Now we define the relative angles as

$$\begin{aligned} &\phi_i = \theta_{i+1} - \theta_i \quad i = 1, \ldots, n-1 \\ &\phi_1 = \theta_2 - \theta_1 \\ &\phi_2 = \theta_3 - \theta_2 \end{aligned}$$

which can be summarized by

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \tag{8.31}$$

The generalized coordinate vector is

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \tag{8.32}$$

with the inverse mapping

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \phi_1 \\ \phi_2 \end{bmatrix} \tag{8.33}$$

From Eq. (8.30)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = 0 \tag{8.34}$$

we get

$$\begin{bmatrix} a_{11} + a_{12} + a_{13} & a_{11} + a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = 0 \tag{8.35}$$

Finally we can write the above equation as follows

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = 0 \tag{8.36}$$

Eq. (8.36) is equivalent to the Pfaffian form $\mathbf{A}(\mathbf{q})\,\dot{\mathbf{q}}$=0.

8.3.2 Investigation of Nonholonomic Criteria

In the previous section we got the kinematic model of a 3-body planar space robot. Now we will investigate whether this system is holonomic or nonholonomic. The procedure of such type of investigation has been already discussed in Chapter 5.

In the next few steps we will derive $\mathbf{G}(\mathbf{q})$, the null space of $\mathbf{A}(\mathbf{q})$.
Let $\dot{\phi}_1 = u_1$ and $\dot{\phi}_2 = u_2$ then we get

$$\begin{aligned} \dot{\mathbf{q}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{\mathbf{A}_{12}u_1 + \mathbf{A}_{13}u_2}{\mathbf{A}_{11}} \\ u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\mathbf{A}_{12}}{\mathbf{A}_{11}} & -\frac{\mathbf{A}_{13}}{\mathbf{A}_{11}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned} \tag{8.37}$$

or in compact form

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q})\,\mathbf{u} \tag{8.38}$$

where (see Appendix G)

$$\begin{aligned} \mathbf{g}_1(\mathbf{q}) &= \begin{bmatrix} g_1(1,1) \\ g_1(2,1) \\ g_1(3,1) \end{bmatrix} = \begin{bmatrix} -\frac{\mathbf{A}_{12}}{\mathbf{A}_{11}} \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{b_{22}+c_1\cos\phi_1+c_2\cos(\phi_1+\phi_2)+b_{22}+2c_3\cos\phi_2+b_{33}}{b_{11}+2c_1\cos\phi_1+2c_2\cos(\phi_1+\phi_2)+b_{22}+2c_3\cos\phi_2+b_{33}} \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}\mathbf{g}_2(\mathbf{q}) &= \begin{bmatrix} g_2(1,1) \\ g_2(2,1) \\ g_2(3,1) \end{bmatrix} = \begin{bmatrix} -\frac{\mathbf{A}_{13}}{\mathbf{A}_{11}} \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{b_{33}+c_2\cos(\phi_1+\phi_2)+c_3\cos\phi_2}{b_{11}+2c_1\cos\phi_1+2c_2\cos(\phi_1+\phi_2)+b_{22}+2c_3\cos\phi_2+b_{33}} \\ 1 \\ 0 \end{bmatrix}\end{aligned}$$

And we will get $\mathbf{g}_3(\mathbf{q})$ by applying Lie bracket operation (see Appendix G)

$$\begin{aligned}\mathbf{g}_3(\mathbf{q}) &= \begin{bmatrix} g_3(1,1) \\ g_3(2,1) \\ g_3(3,1) \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1(\mathbf{q}) & \mathbf{g}_2(\mathbf{q}) \end{bmatrix} \\ &= \frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{q}}\mathbf{g}_2 \end{aligned} \tag{8.39}$$

where

$$\begin{aligned}\mathbf{g}_3(1,1) &= \overline{a} + \overline{b} + \overline{c} - \overline{d} \\ \mathbf{g}_3(2,1) &= 0 \\ \mathbf{g}_3(3,1) &= 0\end{aligned}$$

where

$$\begin{aligned}\overline{a} &= \frac{c_2\sin(\phi_1+\phi_2)}{b_{11}+2c_1\cos\phi_1+2c_2\cos(\phi_1+\phi_2)+b_{22}+2c_3\cos\phi_2+b_{33}} \\ b_1 &= (-2c_1\sin\phi_1 - 2c_2\sin(\phi_1+\phi_2)) \\ \overline{b} &= \frac{(b_{33}+c_2\cos(\phi_1+\phi_2)+c_3\cos\phi_2)\,b_1}{(b_{11}+2c_1\cos\phi_1+2c_2\cos(\phi_1+\phi_2)+b_{22}+2c_3\cos\phi_2+b_{33})^2} \\ \overline{c} &= \frac{-2c_3\sin\phi_2 - c_2\sin(\phi_1+\phi_2)}{b_{11}+2c_1\cos\phi_1+2c_2\cos(\phi_1+\phi_2)+b_{22}+2c_3\cos\phi_2+b_{33}} \\ d_1 &= (b_{22}+c_1\cos\phi_1+c_2\cos(\phi_1+\phi_2)+2c_3\cos\phi_2+b_{33}) \\ d_2 &= (-2c_2\sin(\phi_1+\phi_2)-2c_3\sin\phi_2) \\ \overline{d} &= \frac{(d_1 d_2)}{(b_{11}+2c_1\cos\phi_1+2c_2\cos(\phi_1+\phi_2)+b_{22}+2c_3\cos\phi_2+b_{33})^2}\end{aligned}$$

We obtain the following accessibility distribution matrix:

$$\mathbf{G}'(\mathbf{q}) = \begin{bmatrix} g_1(1,1) & g_2(1,1) & g_3(1,1) \\ g_1(2,1) & g_2(2,1) & g_3(2,1) \\ g_1(3,1) & g_2(3,1) & g_3(3,1) \end{bmatrix} \tag{8.40}$$

We get the Rank of $\mathbf{G}'(\mathbf{q}) = 3$ that means these three columns are linearly independent. We can write down the accessibility distribution:

$$\Delta_{\mathbf{G}'(\mathbf{q})} = \left\{ \; g_1(\mathbf{q}), \quad g_2(\mathbf{q}), \quad g_3(\mathbf{q}) \; \right\} \tag{8.41}$$

According to the Eq. (8.41) we get $\dim \Delta_{\mathbf{G}(\mathbf{q})} = 3 = n$ and then finally we can remark that the Three-body planar space robot system is completely nonholonomic (see Chapter 5).

Table 8.1: Different cases for the Free-floating planar space robot(3-body)

Case	l_1, l_2, l_3 (m)	d_1, d_2, d_3 (m)	m_1, m_2, m_3 (kg)	J_1, J_2, J_3 (kg m^2)	t_s (sec)
1	$\begin{bmatrix}5\\10\\15\end{bmatrix}$	$\begin{bmatrix}2.5\\5.0\\7.5\end{bmatrix}$	$\begin{bmatrix}10\\10\\10\end{bmatrix}$	$\begin{bmatrix}50\\50\\50\end{bmatrix}$	0.1
2	$\begin{bmatrix}5\\10\\15\end{bmatrix}$	$\begin{bmatrix}2.5\\5.0\\7.5\end{bmatrix}$	$\begin{bmatrix}10\\10\\10\end{bmatrix}$	$\begin{bmatrix}100\\100\\100\end{bmatrix}$	0.1
3	$\begin{bmatrix}5\\10\\15\end{bmatrix}$	$\begin{bmatrix}2.5\\5.0\\7.5\end{bmatrix}$	$\begin{bmatrix}20\\20\\20\end{bmatrix}$	$\begin{bmatrix}50\\50\\50\end{bmatrix}$	0.1
4	$\begin{bmatrix}5\\10\\15\end{bmatrix}$	$\begin{bmatrix}4.75\\2.5\\3.75\end{bmatrix}$	$\begin{bmatrix}10\\10\\10\end{bmatrix}$	$\begin{bmatrix}50\\50\\50\end{bmatrix}$	0.1

8.3.3 Simulation Results

In this section, we will simulate the kinematic model of the system using $\mathbf{u}=\begin{pmatrix}\omega_1 & \omega_2\end{pmatrix}^T$ where $\omega_1=\omega_2=1$ rad/sec for different cases (Table 8.1) to investigate the the following characteristics in our system model:

1. Angular momentum of the system
2. Position of the system COM

According to our system model we expect the angular momentum of the system will be constant as well as the position of the system COM.
We will consider the same initial conditions, $\mathbf{q}_0 = \begin{pmatrix}\theta_1 & \phi_2 & \phi_2\end{pmatrix}_0 = \begin{pmatrix}\frac{\pi}{2} & -\frac{\pi}{2} & \frac{\pi}{2}\end{pmatrix}$ for all cases but the length of the body and their own COM could vary according to the cases (Table 8.1).

Case 2 we take the value of the inertia property is twitch than Case 1. But Case 3 we change the mass of the bodies twitch than Case 1 and finally in Case 4 we alter only the position of the center of mass of the corresponding bodies half of the Case 1.

Figure 8.9 shows the initial configurations of the planar space robot (3 body) for case 1, case 2 and case 3. But in case 4, we find the different initial configuration compare to the previous cases (Figure 8.15). From Figure 8.10 to Figure 8.13 show the variation of the system center of mass for four cases and we find that for each cases the system center of mass remain constant at which match with our system model.

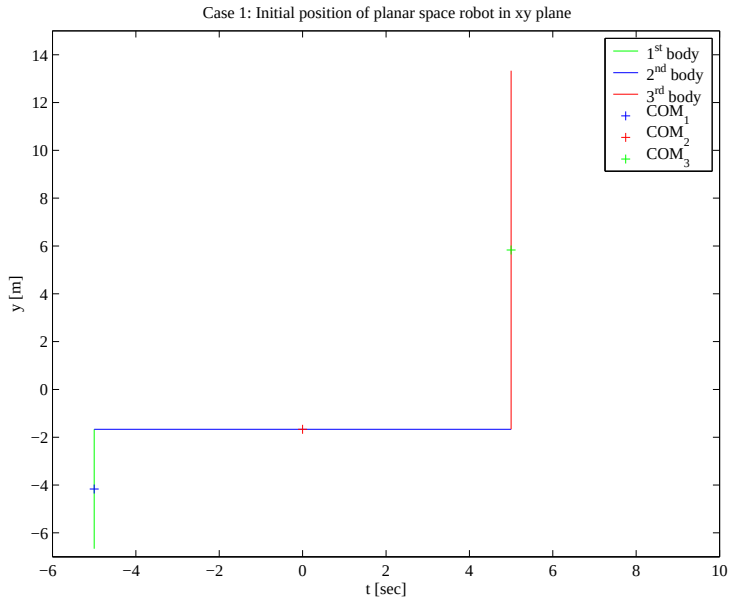

Figure 8.9: Planar Space Robot (3 body) at initial position (For case 1,2,3)

Figure 8.14 shows the angular momentum of the system for four cases and each cases we find that the angular moment remains constant over time in the scale of 10^{-13}.

Figure 8.15 illustrates the trajectory of the system for each cases and we find for case 4 that the system initial configuration is not the same position as compared to previous cases.

From Figure 8.10 to Figure 8.15 we simulate the model up to t=2π sec. But now we will simulate the model $t \geq 2\pi$ sec for further investigation of the system behavior.

As we know the nonholonomic system can switch to a holonomic one. Here we will investigate if there exists any possibility for any geometric properties of the planar space robot (3 body) to exhibit the holonomic behavior. For this reason we simulate the model (Case 4) up to $t = 4\pi$ sec.

Figure 8.16 shows at $t = 6.5$ sec the internal configurations reach at $\mathbf{q}_0 = \begin{pmatrix} \theta_1 & \phi_2 & \phi_2 \end{pmatrix}_0$ $= \begin{pmatrix} \frac{\pi}{2} & -\frac{\pi}{2} & \frac{\pi}{2} \end{pmatrix}$ (approximately). Figure 8.17 shows that the system (Case 4) has tendency to reach its original posture. Figure 8.18 shows the final orientation of the system (Case 1) after $t = 8\pi$ sec. So we may conclude that with a special geometric properties the nonholonomic system can switch to a holonomic one.

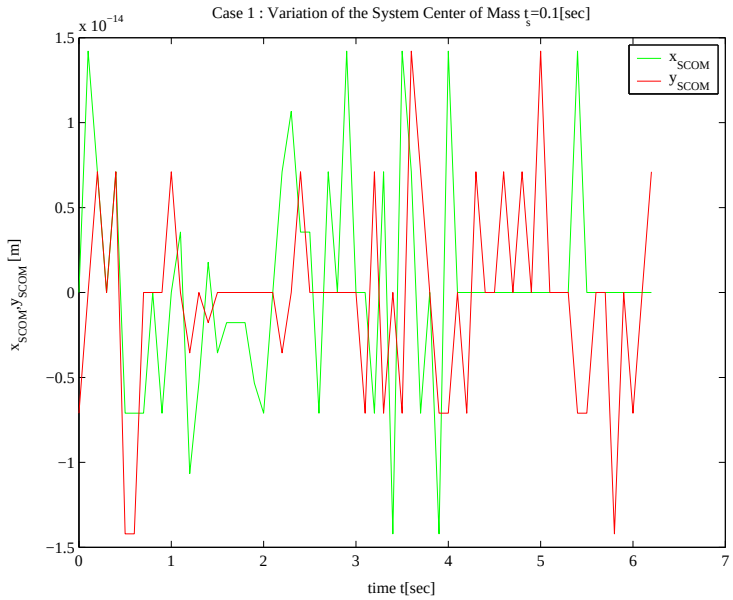

Figure 8.10: Variation of the system center of mass over time (Case 1)

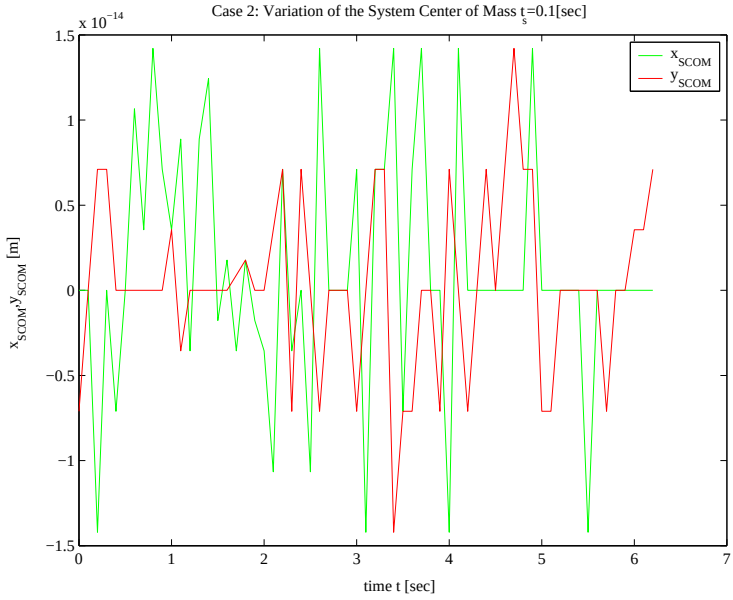

Figure 8.11: Variation of the system Center of mass over time (Case 2)

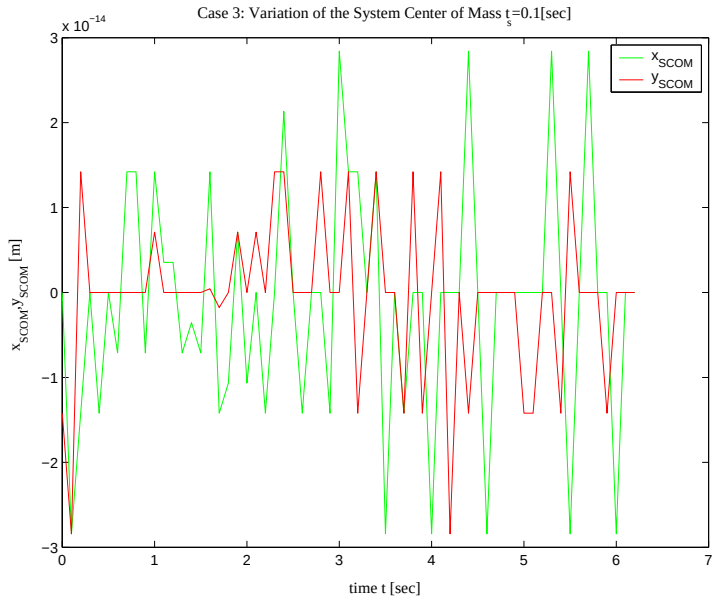

Figure 8.12: Variation of the system center of mass over time (Case 3)

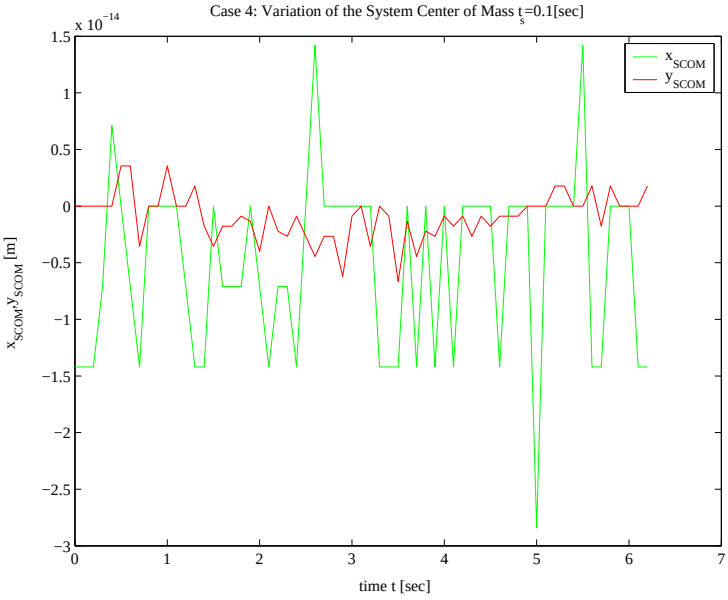

Figure 8.13: Variation of the system center of mass over time (Case 4)

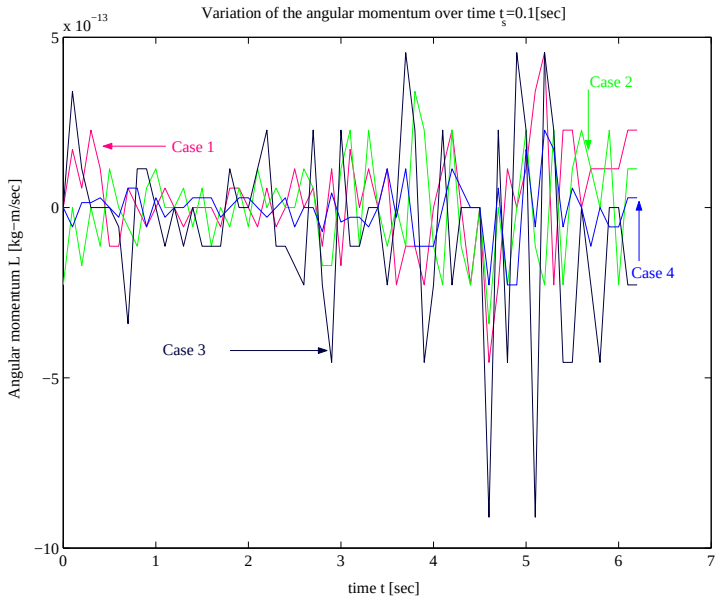

Figure 8.14: Variation of the angular momentum over time for different cases

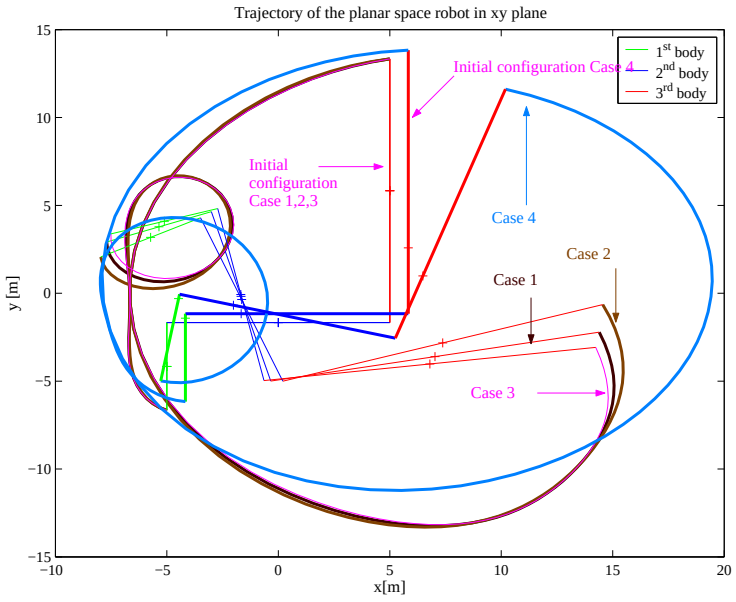

Figure 8.15: Trajectory of the system for different cases

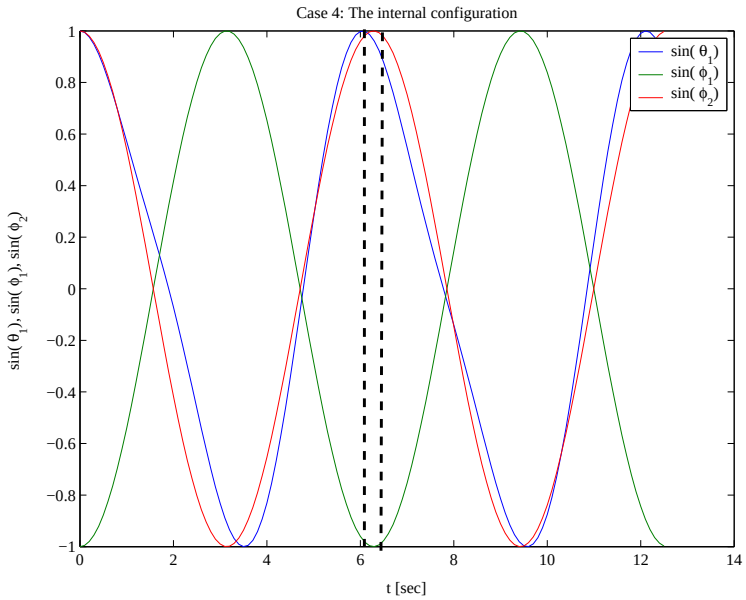

Figure 8.16: Internal configurations of the planar space robot (3-body)

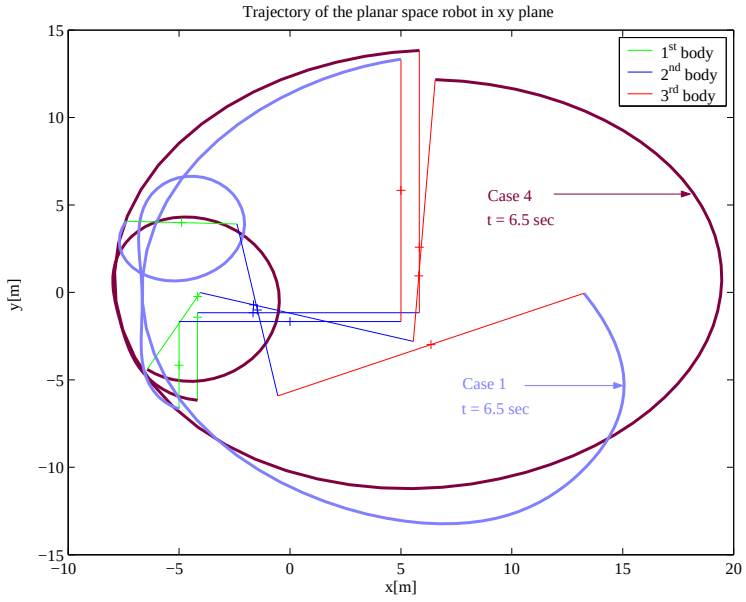

Figure 8.17: of the planar space robot (3-body) (up to t = 6.5 sec)

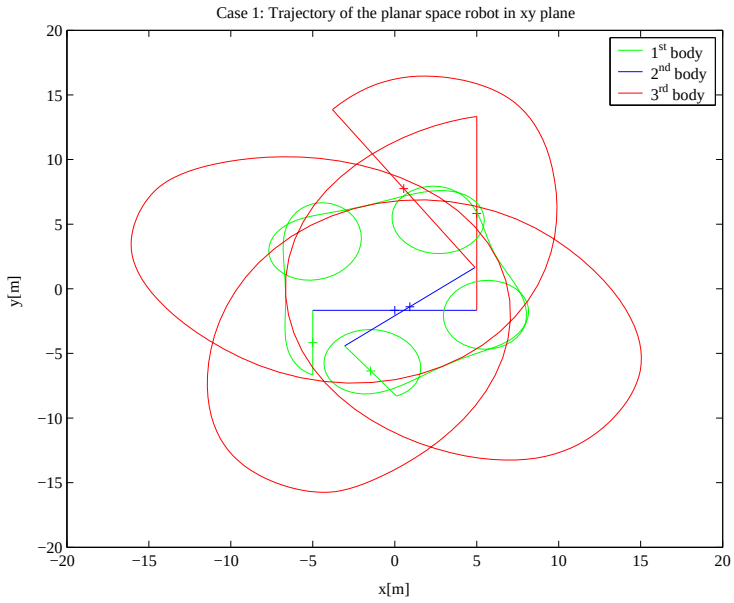

Figure 8.18: Trajectory of the (3-body) planar space robot (Case 1) (up to t = 25 sec)

Chapter 9

Free-Floating Space Robot in 3D Motion (2-Body)

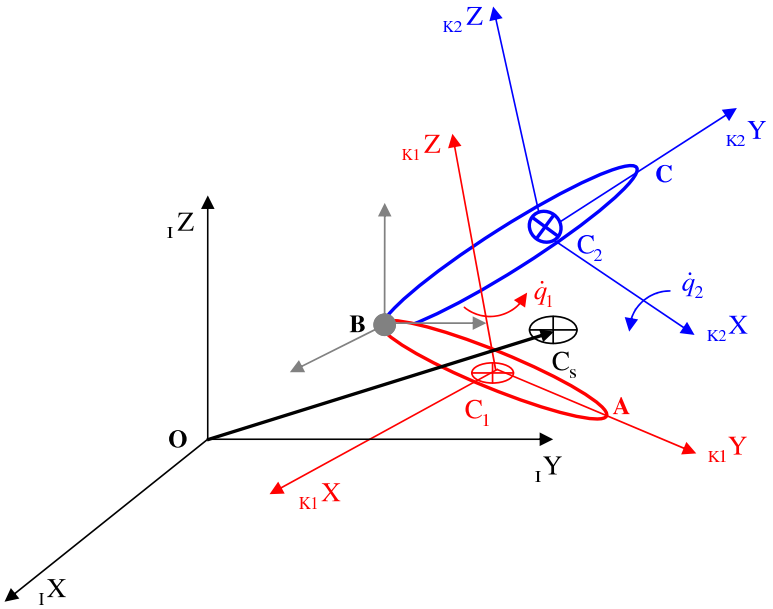

Figure 9.1: Two-body robot in 3D space

9.1 System Description

Here the system consists of two bodies AB (1^{st} body) and BC (2^{nd} body). They are joint at point B (revolute joint) which consists of two axises where point B and B-frame are attached with the 1^{st} body. The one axis represent the 1^{st} joint and the other axis represent the 2^{nd} joint. We also consider the two body-fixed reference frames of K1 and K2 for 1^{st} and 2^{nd} body at their own center of mass (COM) at C_1 and C_2 respectively. We assume that the position of the center of mass of the 1^{st} body and 2^{nd} body lie on their own Y-axis and the system center of mass is located at point C_s. Initially, we assume that the two bodies lie on

each other and are floating in space with initial angular and linear velocities. After then the 1^{st} joint rotates around ${}_{K1}Z$, at this moment two bodies lie on the same plane as described in the previous Chapter in case of the planar space robot. After then the 2^{nd} joint rotates around ${}_{K2}X$-axis, at this moment the 2^{nd} body comes out of the plane (Figure 9.1).

9.2 System Model

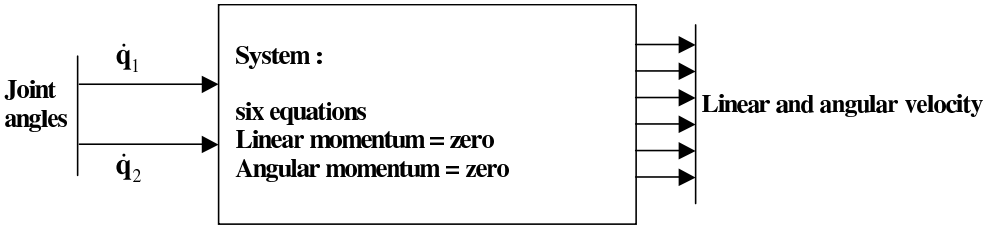

Figure 9.2: Model of the system

If we assume that the system is in rest in the initial state, that means the linear and angular momentum are zero then we will obtain the new linear and angular velocity of the system in the inertia frame I-frame to change its joint angles. The position and orientation of the body depends on its inertia property which change with its configuration. In our particular case we can model our system as shown in Figure 9.2 where we have six unknowns (three linear and three angular velocities of the system) with two relative joint angles $\dot{q_1}$ and $\dot{q_2}$ as input.

9.3 Constraint Equations and Kinematic Model

The following notation are used in the derivation of the constraint equations:

${}_I\omega_b$	Angular velocity of the 1^{st} body with respect to I-frame
${}_I\mathbf{v}_b$	Linear velocity of the 1^{st} body with respect to I-frame
${}_{K1}\omega_1$	Relative angular velocity of the 1^{st} joint with respect to K1-frame
${}_{K2}\omega_2$	Relative angular velocity of the 2^{nd} joint with respect to K2-frame
${}_I\mathbf{v}_{com1}$	Linear velocity of the 1^{st} body at its COM with respect to I-frame
${}_I\mathbf{v}_{com2}$	Linear velocity of the 2^{nd} body at its COM with respect to I-frame
${}_I\mathbf{P}_1$	Linear momentum of the 1^{st} body with respect to I-frame
${}_I\mathbf{P}_2$	Linear momentum of the 2^{nd} body with respect to I-frame
${}_I\mathbf{L}_1$	Angular momentum of the 1^{st} body with respect to I-frame
${}_I\mathbf{L}_2$	Angular momentum of the 2^{nd} body with respect to I-frame
${}_{K1}\mathbf{L}_1$	Angular momentum of the 1^{st} body with respect to K1-frame

${}_{K2}\mathbf{L}_1$	Angular momentum of the 2^{nd} body with respect to K2-frame
${}_{K1}\mathbf{r}_{B_com1}$	Distance from B point to its COM with respect to K1-frame
${}_{K2}\mathbf{r}_{B_com2}$	Distance from B point to its COM with respect to K2-frame
${}_{K1}\mathbf{r}_{B_0} = \mathbf{r}$	Distance from B point to the origin of the I-frame
${}_{K1}\mathbf{r}_{0_COM1}$	Position vector of COM of the 1^{st} body
${}_{K2}\mathbf{r}_{0_COM2}$	Position vector of COM of the 2^{nd} body
${}_{I}\mathbf{r}_{0_C_s}$	Position vector of system center of mass
m_1	Mass of the 1^{st} body
m_2	Mass of the 2^{nd} body
${}_{K1}\mathbf{J}_{COM1} = \mathbf{J}_1$	Inertia tensor of the 1^{st} body with respect to its COM in K1-frame
${}_{K2}\mathbf{J}_{COM2} = \mathbf{J}_2$	Inertia tensor of the 2^{nd} body with respect to its COM in K2-frame

The following quantities are given below:

$$
\begin{aligned}
{}_{K1}\mathbf{r}_{B_com1} = \mathbf{r}_1 &= \begin{pmatrix} r_{1x} & r_{1y} & r_{1z} \end{pmatrix}^T = \begin{pmatrix} 0 & d_1 & 0 \end{pmatrix}^T \\
{}_{K2}\mathbf{r}_{B_com2} = \mathbf{r}_2 &= \begin{pmatrix} r_{2x} & r_{2y} & r_{2z} \end{pmatrix}^T = \begin{pmatrix} 0 & d_2 & 0 \end{pmatrix}^T \\
{}_{K2}\omega_1 &= \begin{pmatrix} 0 & 0 & \dot{q}_1 \end{pmatrix}^T \\
{}_{K2}\omega_2 &= \begin{pmatrix} \dot{q}_2 & 0 & 0 \end{pmatrix}^T
\end{aligned}
$$

The corresponding tilde operators are

$$
\tilde{\mathbf{r}}_1 = \begin{bmatrix} 0 & -r_{1z} & r_{1y} \\ r_{1z} & 0 & -r_{1x} \\ -r_{1y} & r_{1x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & d_1 \\ 0 & 0 & 0 \\ -d_1 & 0 & 0 \end{bmatrix}
$$

$$
\tilde{\mathbf{r}}_2 = \begin{bmatrix} 0 & -r_{2z} & r_{2y} \\ r_{2z} & 0 & -r_{2x} \\ -r_{2y} & r_{2x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & d_2 \\ 0 & 0 & 0 \\ -d_2 & 0 & 0 \end{bmatrix}
$$

and their products

$$
\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_1 = \begin{bmatrix} r_{1y}^2 + r_{1z}^2 & -r_{1x}r_{1y} & -r_{1x}r_{1z} \\ -r_{1x}r_{1y} & r_{1z}^2 + r_{1x}^2 & -r_{1y}r_{1z} \\ -r_{1x}r_{1z} & -r_{1y}r_{1z} & r_{1y}^2 + r_{1x}^2 \end{bmatrix} = \begin{bmatrix} d_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_1^2 \end{bmatrix}
$$

$$
\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2 = \begin{bmatrix} r_{2y}^2 + r_{2z}^2 & -r_{2x}r_{2y} & -r_{2x}r_{2z} \\ -r_{2x}r_{2y} & r_{2z}^2 + r_{2x}^2 & -r_{2y}r_{2z} \\ -r_{2x}r_{2z} & -r_{2y}r_{2z} & r_{2y}^2 + r_{2x}^2 \end{bmatrix} = \begin{bmatrix} d_2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_2^2 \end{bmatrix}
$$

We take the following rotational matrix

$$
\begin{aligned}
\mathbf{R}_{K1K2} &= \mathbf{R}_X(q_2)\mathbf{R}_Z(q_1) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_2 & -\sin q_2 \\ 0 & \sin q_2 & \cos q_2 \end{bmatrix} \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

The angular positions of the B-frame are described by α,β, and γ and we consider the B-frame changes its orientation over time with respect to I-frame. We also consider the three rotations taking place about fixed reference frame, I-frame by the following order:

$$\begin{aligned}\mathbf{R}_{IK1} &= \mathbf{R}_Z(\gamma)\mathbf{R}_Y(\beta)\mathbf{R}_X(\alpha) \\ &= \begin{bmatrix}\cos\gamma & -\sin\gamma & 0\\ \sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix} \\ &= \begin{bmatrix}\cos\gamma\cos\beta & \sin\alpha\sin\beta\cos\gamma-\cos\alpha\sin\gamma & \cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma\\ \sin\gamma\cos\beta & \sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\gamma\sin\beta\cos\alpha-\sin\alpha\cos\gamma\\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha\end{bmatrix}\end{aligned}$$

Now we obtain the linear velocities at COM with respect to inertia frame for 1^{st} and 2^{nd} body respectively :

$$\begin{aligned}{}_I\mathbf{v}_{com1} &= {}_I\mathbf{v}_b +_I \omega_b \times_I \mathbf{r}_{B_com1} \\ &= {}_I\mathbf{v}_b +_I \omega_b \times \mathbf{R}_{IK1\ K1}\mathbf{r}_{B_com1}\end{aligned} \tag{9.1}$$

and

$$\begin{aligned}{}_I\mathbf{v}_{com2} &= {}_I\mathbf{v}_b +_I \omega_2^{abs} \times_I \mathbf{r}_{B_com2} \\ &= {}_I\mathbf{v}_b + ({}_I\omega_b +_I \omega_1 +_I \omega_2) \times {}_I\mathbf{r}_{B_com2} \\ &= {}_I\mathbf{v}_b + ({}_I\omega_b + \mathbf{R}_{IK1\ K1}\omega_1 \\ &+ \mathbf{R}_{IK1}\mathbf{R}_{K1K2\ K2}\omega_2) \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2\ K2}\mathbf{r}_{B_com2}\end{aligned} \tag{9.2}$$

We obtained the corresponding linear momentum with respect to inertia frame

$${}_I\mathbf{P}_1 = m_1\ {}_I\mathbf{v}_{com1} \tag{9.3}$$

$${}_I\mathbf{P}_2 = m_2\ {}_I\mathbf{v}_{com2} \tag{9.4}$$

So we get the total linear momentum in inertia frame by adding Eq. (9.3) and Eq. (9.4)

$${}_I\mathbf{P} = {}_I\mathbf{P}_1 +_I \mathbf{P}_2 \tag{9.5}$$

Now we calculate the angular momentum in K1 and K2 frame:

$$\begin{aligned}{}_{K1}\mathbf{L}_1 &= ,_{K1}\mathbf{J}_{COM1\ K1}\omega_b + ({}_{K1}\mathbf{r}_{B_com1} + {}_{K1}\mathbf{r}_{B_0}) \times (m_1\ {}_{K1}\mathbf{v}_{com1}) \\ &= {}_{K1}\mathbf{J}_{COM1}\mathbf{R}_{K1I}\omega_b + ({}_{K1}\mathbf{r}_{B_com1} + {}_{K1}\mathbf{r}_{B_0}) \times (m_1\ {}_{K1}\mathbf{v}_{com1})\end{aligned} \tag{9.6}$$

$$\begin{aligned}{}_{K2}\mathbf{L}_2 &= {}_{K2}\mathbf{J}_{COM2\ K2}\omega_b^{abs} + ({}_{K2}\mathbf{r}_{B_com2} + {}_{K2}\mathbf{r}_{B_0}) \times (m_2\ {}_{K2}\mathbf{v}_{com2}) \\ &= {}_{K2}\mathbf{J}_{COM2}({}_{K2}\omega_b + {}_{K2}\omega_1 + {}_{K2}\omega_2) \\ &+ ({}_{K2}\mathbf{r}_{B_com2} + {}_{K2}\mathbf{r}_{B_0}) \times (m_2\ {}_{K2}\mathbf{v}_{com2})\end{aligned} \tag{9.7}$$

Then we get the total angular momentum with respect to I-frame:

$$\begin{aligned}{}_I\mathbf{L} &= {}_I\mathbf{L}_{K1} +_I \mathbf{L}_{K2} \\ &= \mathbf{R}_{IK1\ K1}\mathbf{L}_1 + \mathbf{R}_{IK1}\mathbf{R}_{K1K2\ K2}\mathbf{L}_1\end{aligned} \tag{9.8}$$

From Eq. (9.5) we get total linear momentum with respect to I-frame:

$$\begin{aligned}
{}_I\mathbf{P} &= {}_I\mathbf{P}_1 + {}_I\mathbf{P}_2 \\
&= (m_1 + m_2)\,{}_I\mathbf{v}_b - (m_1\mathbf{R}_{IK1}\tilde{\mathbf{r}}_1 + m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2)\,{}_I\omega_b \\
&\quad - m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2\mathbf{R}_{IK1}\dot{\mathbf{q}}_1 - m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\dot{\mathbf{q}}_2
\end{aligned} \tag{9.9}$$

So we can write down the above equation in the following form

$$\mathbf{P} = \mathbf{a}_{11}\mathbf{v}_b + \mathbf{a}_{12}\omega_b + \mathbf{b}_{11}\dot{\mathbf{q}}_1 + \mathbf{b}_{12}\dot{\mathbf{q}}_2 \tag{9.10}$$

where

$$\mathbf{a}_{11} = \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{1}^{32} & a_{11}^{33} \end{bmatrix} = \begin{bmatrix} m_1+m_2 & 0 & 0 \\ 0 & m_1+m_2 & 0 \\ 0 & 0 & m_1+m_2 \end{bmatrix}$$

$$\mathbf{a}_{12} = \begin{bmatrix} a_{12}^{11} & a_{12}^{12} & a_{12}^{13} \\ a_{12}^{21} & a_{12}^{22} & a_{12}^{23} \\ a_{12}^{31} & a_{12}^{32} & a_{12}^{33} \end{bmatrix} = -(m_1\mathbf{R}_{IK1}\tilde{\mathbf{r}}_1 + m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2)$$

$$\mathbf{b}_{11} = \begin{bmatrix} b_{11}^{11} & b_{11}^{12} & b_{11}^{13} \\ b_{11}^{21} & b_{11}^{22} & b_{11}^{23} \\ b_{11}^{31} & b_{1}^{32} & b_{11}^{33} \end{bmatrix} = -m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2\mathbf{R}_{IK1}$$

$$\mathbf{b}_{12} = \begin{bmatrix} b_{12}^{11} & b_{12}^{12} & b_{12}^{13} \\ b_{12}^{21} & b_{12}^{22} & b_{12}^{23} \\ b_{12}^{31} & b_{12}^{32} & b_{12}^{33} \end{bmatrix} = -m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}$$

The angular momentum of the 1^{st} body with respect to K1-frame is (see Appendix F)

$${}_{K1}\mathbf{L}_1 = m_1\tilde{\mathbf{r}}_1\mathbf{v}_b + (\mathbf{J}_1 - m_1\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_1)\omega_b + \tilde{\mathbf{r}}\cdot\,{}_{K1}\mathbf{P}_1 \tag{9.11}$$

Similarly from Eq. (9.7) we get the angular momentum of the 2^{nd} body with respect to K2-frame. (see Appendix F):

$${}_{K2}\mathbf{L}_2 = m_2\tilde{\mathbf{r}}_2\mathbf{v}_b + (\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2\cdot\tilde{\mathbf{r}}_2)\omega_b + (\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2\cdot\tilde{\mathbf{r}}_2)\omega_2 + \tilde{\mathbf{r}}\cdot\,{}_{K2}\mathbf{P}_2 \tag{9.12}$$

The total angular momentum with respect to I-frame is (see Appendix F)

$$\begin{aligned}
{}_I\mathbf{L}_2 &= \mathbf{R}_{IK1}\,{}_{IK1}L_1 + \mathbf{R}_{IK1}\mathbf{R}_{K1K2,K2}\,\mathbf{L}_1 \\
&= (m_1\tilde{\mathbf{r}}_1 + m_2\tilde{\mathbf{r}}_2)\,{}_I\mathbf{v}_b + ({}_{K1}\mathbf{J}_1 - m_1\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_1 + {}_{K2}\,\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2\cdot\tilde{\mathbf{r}}_2)_I\omega_b \\
&+ ({}_{K2}\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2\cdot\tilde{\mathbf{r}}_2)_I\omega_1 + ({}_{K2}\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2\cdot\tilde{\mathbf{r}}_2)_I\omega_2 + \tilde{\mathbf{r}}\cdot\,{}_I\mathbf{P}
\end{aligned} \tag{9.13}$$

So we can write down the above equation in the following form

$$\mathbf{L} = \mathbf{a}_{21}\mathbf{v}_b + \mathbf{a}_{22}\omega_b + \mathbf{b}_{21}\dot{\mathbf{q}}_1 + \mathbf{b}_{22}\dot{\mathbf{q}}_2 + \tilde{\mathbf{r}}\cdot\mathbf{P} \tag{9.14}$$

where

$$\mathbf{a}_{21} = \begin{bmatrix} a_{21}^{21} & a_{21}^{12} & a_{21}^{13} \\ a_{21}^{21} & a_{21}^{22} & a_{21}^{23} \\ a_{21}^{31} & a_{21}^{32} & a_{21}^{33} \end{bmatrix} = m_1\tilde{\mathbf{r}}_1 + m_2\tilde{\mathbf{r}}_2$$

$$\mathbf{a}_{22} = \begin{bmatrix} a_{22}^{11} & a_{22}^{12} & a_{22}^{13} \\ a_{22}^{21} & a_{22}^{22} & a_{22}^{23} \\ a_{22}^{31} & a_{22}^{32} & a_{22}^{33} \end{bmatrix} = ({}_{K1}\mathbf{J}_1 - m_1\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_1 + {}_{K2}\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2)$$

$$\mathbf{b}_{21} = \begin{bmatrix} b_{21}^{11} & b_{21}^{12} & b_{21}^{13} \\ b_{21}^{21} & b_{21}^{22} & b_{21}^{23} \\ b_{21}^{31} & b_{21}^{32} & b_{21}^{33} \end{bmatrix} = {}_{K2}\,\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2$$

$$\mathbf{b}_{22} = \begin{bmatrix} b_{22}^{11} & b_{22}^{12} & b_{22}^{13} \\ b_{22}^{21} & b_{22}^{22} & b_{22}^{23} \\ b_{22}^{31} & b_{22}^{32} & b_{22}^{33} \end{bmatrix} = {}_{K2}\,\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2$$

Substituting the value of $\mathbf{P}$ into Eq. (9.14) we finally get the two vector equations

$$\mathbf{P} = \mathbf{a}_{11}\mathbf{v}_b + \mathbf{a}_{12}\omega_b + \mathbf{b}_{11}\dot{\mathbf{q}}_1 + \mathbf{b}_{12}\dot{\mathbf{q}}_2 \tag{9.15}$$

$$\begin{aligned} \mathbf{L} = {} & (\mathbf{a}_{21} + \tilde{\mathbf{r}} \cdot \mathbf{a}_{11})\mathbf{v}_b + (\mathbf{a}_{22} + \tilde{\mathbf{r}} \cdot \mathbf{a}_{12})\omega_b \\ & + (\mathbf{b}_{21} + \tilde{\mathbf{r}} \cdot \mathbf{b}_{11})\dot{\mathbf{q}}_1 + (\mathbf{b}_{22} + \tilde{\mathbf{r}} \cdot \mathbf{b}_{12})\dot{\mathbf{q}}_2 \end{aligned} \tag{9.16}$$

We can represent these equations by the following matrix form

$$\begin{aligned} \begin{bmatrix} \mathbf{P} \\ \mathbf{L} \end{bmatrix} = {} & \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} + \tilde{\mathbf{r}} \cdot \mathbf{a}_{11} & \mathbf{a}_{22} + \tilde{\mathbf{r}} \cdot \mathbf{a}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{v}_b \\ \omega_b \end{bmatrix} \\ & + \begin{bmatrix} \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{21} + \tilde{\mathbf{r}} \cdot \mathbf{b}_{11} & \mathbf{b}_{22} + \tilde{\mathbf{r}} \cdot \mathbf{b}_{12} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix} \end{aligned} \tag{9.17}$$

Requiring $\mathbf{P} = \mathbf{0}$ and $\mathbf{L} = \mathbf{0}$ then Eq. (9.17) becomes

$$\begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v}_b \\ \omega_b \end{bmatrix} + \begin{bmatrix} \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{21} & \mathbf{b}_{22} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{9.18}$$

where Eq. (9.14) with $\mathbf{P} = \mathbf{0}$ has been used.
Finally we can represent Eq. (9.18) by the following matrix form:

$$\begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{b}_{21} & \mathbf{b}_{22} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathbf{v}_b \\ \omega_b \end{bmatrix} \\ \\ \begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix} \end{bmatrix} = 0 \tag{9.19}$$

and in compact form

$$\mathbf{A}_{2\times4}(\mathbf{q})\dot{\mathbf{q}}_{2\times4} = 0 \tag{9.20}$$

The associated kinematic model is (see Appendix F)

$$\dot{\mathbf{q}}_{4\times1} = \mathbf{G}_{4\times2}(\mathbf{q})\mathbf{A}_{2\times1} \tag{9.21}$$

where (see Appendix F)

$$\mathbf{G}_{4\times2}(\mathbf{q}) = \begin{bmatrix} \frac{\mathbf{a}_{12}\mathbf{b}_{21}-\mathbf{b}_{11}\mathbf{a}_{22}}{\mathbf{a}_{11}\mathbf{b}_{22}-\mathbf{b}_{21}\mathbf{a}_{12}} & \frac{\mathbf{a}_{12}\mathbf{b}_{22}-\mathbf{b}_{12}\mathbf{a}_{22}}{\mathbf{a}_{11}\mathbf{a}_{22}-\mathbf{a}_{21}\mathbf{a}_{12}} \\ -\frac{-\mathbf{a}_{21}\mathbf{b}_{11}+\mathbf{a}_{11}\mathbf{b}_{21}}{\mathbf{a}_{11}\mathbf{a}_{22}-\mathbf{a}_{21}\mathbf{a}_{12}} & -\frac{\mathbf{a}_{11}\mathbf{b}_{22}-\mathbf{a}_{21}\mathbf{b}_{12}}{\mathbf{a}_{11}\mathbf{a}_{22}-\mathbf{a}_{21}\mathbf{a}_{12}} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Now we expand Eq. (9.21) and we get the following kinematic model as we discussed before:

$$\dot{\mathbf{q}} = (v_{bx} \quad v_{by} \quad v_{bz} \quad \omega_{bx} \quad \omega_{by} \quad \omega_{bz} \quad \omega_{1x} \quad \omega_{1y} \quad \omega_{1z} \quad \omega_{2x} \quad \omega_{2y} \quad \omega_{2z})^T$$

and

$$\begin{aligned} \mathbf{u} &= (\omega_{1x} \quad \omega_{1y} \quad \omega_{1z} \quad \omega_{2x} \quad \omega_{2y} \quad \omega_{2z})^T \\ &= (\dot{q}_{1x} \quad \dot{q}_{1y} \quad \dot{q}_{1z} \quad \dot{q}_{2x} \quad \dot{q}_{2y} \quad \dot{q}_{2z})^T \end{aligned}$$

So we can rewrite Eq. (9.21)

$$\begin{aligned} \dot{\mathbf{q}}_{12\times1} &= \mathbf{G}_{12\times6}(\mathbf{q})\mathbf{u}_{6\times1} \\ &= \text{span}\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4, \mathbf{g}_5, \mathbf{g}_6\}\mathbf{u}_{6\times1} \end{aligned} \tag{9.22}$$

But in our particular case

$$\mathbf{u} = (0 \quad 0 \quad \dot{q}_1 \quad \dot{q}_2 \quad 0 \quad 0)^T$$

where $\dot{q}_1 = \omega_1$, $\dot{q}_2 = \omega_2$, $\mathbf{u}_1 = (0 \quad 0 \quad \omega_1)^T$ and $\mathbf{u}_2 = (\omega_2 \quad 0 \quad 0)^T$ So our associated kinematic model (see Appendix F) is condensed to

$$\dot{\mathbf{q}}_{8\times1} = \mathbf{G}'_{8\times2}(\mathbf{q})\mathbf{u}_{2\times1} = \text{span}\{\mathbf{g}'_3, \mathbf{g}'_4\}\mathbf{u}_{2\times1} \tag{9.23}$$

with

$$\dot{\mathbf{q}} = (v_{bx} \quad v_{by} \quad v_{bz} \quad \omega_{bx} \quad \omega_{by} \quad \omega_{bz} \quad 0 \quad 0 \quad \omega_{1z} \quad \omega_{2x} \quad 0 \quad 0)^T$$

and

$$\begin{aligned} \mathbf{u} &= (0 \quad 0 \quad \omega_{1z} \quad \omega_{2x} \quad 0 \quad 0)^T \\ &= (0 \quad 0 \quad \dot{q}_{1z} \quad \dot{q}_{2x} \quad 0 \quad 0)^T \end{aligned}$$

9.3.1 Simulation Results

In this section, we conduct the case studies to show the following characteristics in our system model for different sampling times and different cases (Table 9.1).

1. Position of the system center of mass (COM)

Table 9.1: Different cases for Free-floating space robot (3D motion)

Case	u_1 (rad/sec)	u_2 (rad/sec)	$\mathbf{J}_1$ (kg m^2)	$\mathbf{J}_2$ (kg m^2)	m_1 (kg)	m_2 (kg)	t_s (sec)	d_1 (m)	d_2 (m)
1	$\begin{bmatrix}0\\0\\\omega_1\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}100&0&0\\0&100&0\\0&0&100\end{bmatrix}$	$\mathbf{J}_1$	10	m_1	0.1 0.01 0.001	1	1.5
2	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}\omega_2\\0\\0\end{bmatrix}$	$\begin{bmatrix}100&0&0\\0&100&0\\0&0&100\end{bmatrix}$	$\mathbf{J}_1$	10	m_1	0.1 0.01 0.001	1	1.5
3	$\begin{bmatrix}0\\0\\\omega_1\end{bmatrix}$	$\begin{bmatrix}\omega_2\\0\\0\end{bmatrix}$	$\begin{bmatrix}100&0&0\\0&100&0\\0&0&100\end{bmatrix}$	$\mathbf{J}_1$	10	m_1	0.1 0.01 0.001	1	1.5

2. Linear momentum of the system

3. Angular momentum of the system

According to our model the linear and angular momentum will be constant as well as the position of the system COM. The angular and linear velocity of the base B also will be shown.

In Chapter 8 we consider the inertia frame, I-frame at the system center of mass i.e., the system COM always at (0,0). But here we will obtain the system COM by the following equation

$$\mathbf{r}_{0_C_s} = \frac{m_1\mathbf{r}_{0_C_1} + m_2\mathbf{r}_{0_C_2}}{m_1 + m_2} = c \tag{9.24}$$

and the value of the constant c depends on the chosen initial conditions. As in all cases the values of m_1, m_2, d_1 and d_2 do not change (see Table 9.1) so we expect that the system COM always remain at $\mathbf{r}_{0_C_s} = (0 \quad 1.25 \quad 0)^T\ m$.

We run the simulation model with constant speeds, i.e., $\omega_1 = \omega_2 = 1$ rad/sec upto $t = 2\pi$ sec and take the same initial conditions $\dot{\mathbf{q}} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T$ for all cases.

In case 1, as we take $u_1 = (0 \quad 0 \quad \omega_1)^T$ that means we apply the input only at ${}_{K1}Z$-axis and we expect that the angular velocity in x-axis and y-axis are to be zero (see Figure 9.7) which shows that the space robot rotates on the xy plane as compare to Chapter 8. As a result we also expceted that the linear velocity of z-component will be always zero (see Figure 9.6).

Similarly in case 2, as we take $u_2 = (\omega_2 \quad 0 \quad 0)^T$ that means we apply the input only at ${}_{K2}X$-axis and we expect the angular velocity in y-axis and z-axis is zero (see Figure 9.12)

which shows that the space robot lies on the yz plane. As a result we also expcet the linear velocity of x-component will be always zero (see Figure 9.11).

But in case 3, as we take $u_1 = (0 \quad 0 \quad \omega_1)^T$ and $u_2 = (\omega_2 \quad 0 \quad 0)^T$ that means we apply the input both at ${}_{K2}X$-axis and ${}_{K1}Z$-axis and we expect the angular velocity in y-axis is zero (see Figure 9.17). Here we also expcet the non-zero values of linear velocity in x-componant, y-component and z-component (see Figure 9.16).

Figure 9.3, Figure 9.4 and Figure 9.5 are showing the system center of mass, linear momentum and angular momentum of the system for case 1 with sampling time $t_s = 0.1$ sec.

Figure 9.8, Figure 9.9 and Figure 9.10 are showing the system center of mass, linear momentum and angular momentum of the system for case 2 with sampling time $t_s = 0.1$ sec.

Figure 9.13, Figure 9.14 and Figure 9.15 are showing the system center of mass, linear momentum and angular momentum of the system for case 3 with sampling time $t_s = 0.1$ sec.

For every cases we find that the value of R_{cy} varies slightly over time starts from 1.25 m as expected, and after $t = 2\pi$ sec the COM returns its initial position (apporximately) for case 1 and case 2 (Figure 9.3, Figure 9.8). But in case 3 the system COM varies over time.

In case 1, we apply the input only in the ${}_{K1}Z$-axis, and we find that the z-component of the linear momentum P_z, the x-component of the angular momentum L_x, and the y-component of the angular momentum L_y are always zero as expected. But we find P_x, P_y varies over time in the scale of $10^{-14}m$ (Figure 9.4) and L_z varies over time in the scale of $10^{-14}m$ (Figure 9.5).

Similarly in case 2, we apply the input only in the ${}_{K2}X$-axis, and we find that the x-component of the linear momentum P_x, the y-component of the angular momentum L_y, and the z-component of the angular momentum L_z are always zero as expected. But we find P_y, P_z varies over time in the scale of $10^{-14}m$ (Figure 9.9) and L_x varies over time in the scale of $10^{-14}m$ (Figure 9.10).

In case 3, we find that the linear and angular momentum varies over time in the scale of $10^{-14}m$ (Figure 9.14, Figure 9.15).

Now we may conclude that according to our system model the linear and angular momentum of the system should be zero and we get the same results for each case as expected. And the system COM, as we expected, always remain at $\mathbf{r}_{0_C} = (0 \quad 1.25 \quad 0)^T$ m. We get this result for case 1 and case 2 , but we find that in case 3 the system COM varies over time. We find almost the same system behaviour for others sampling time thats why other simulation results are not shown in this paper.

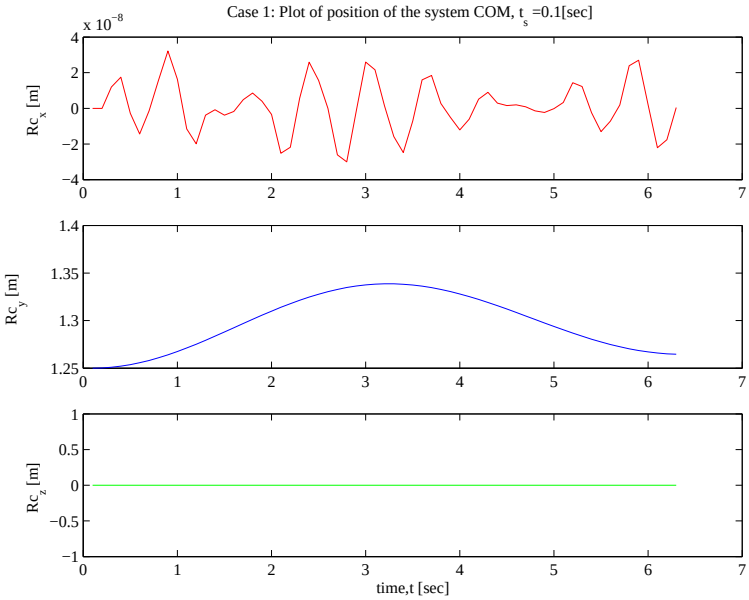

Figure 9.3: System center of mass (Case 1)

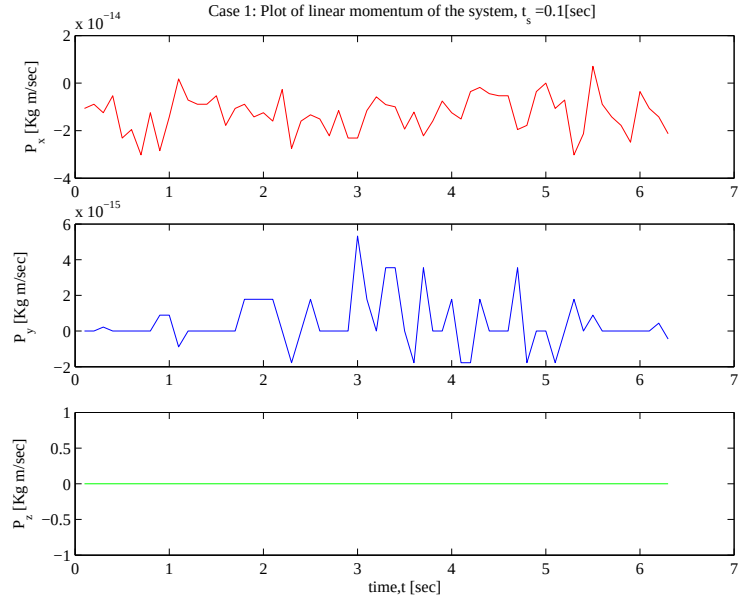

Figure 9.4: Linear momentum of the system (Case 1)

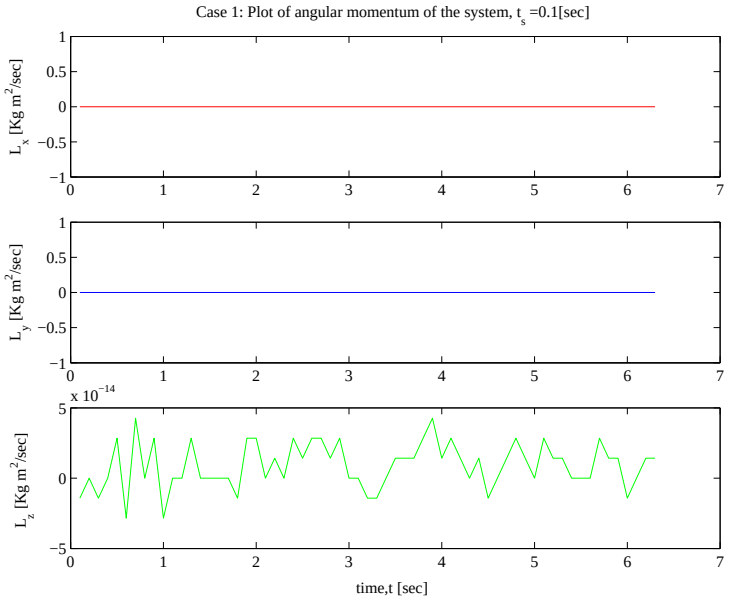

Figure 9.5: Angular momentum of the system (Case 1)

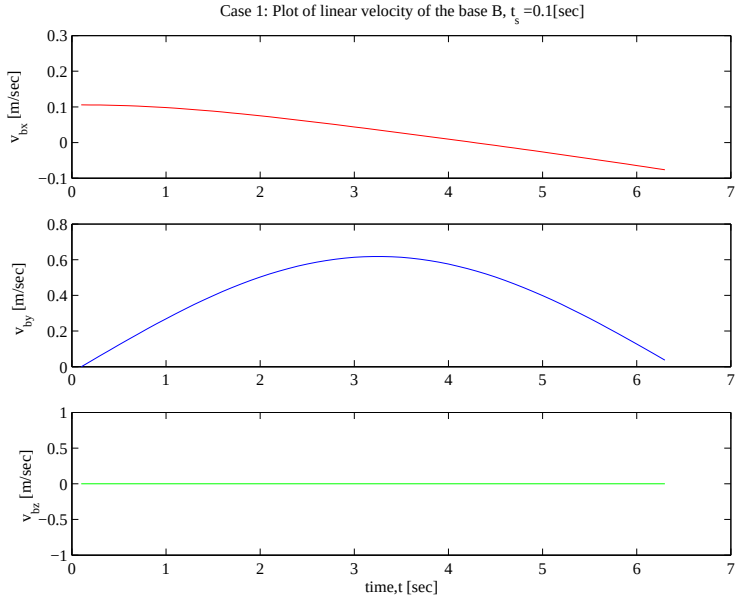

Figure 9.6: Linear velocity of the base, B (Case 1)

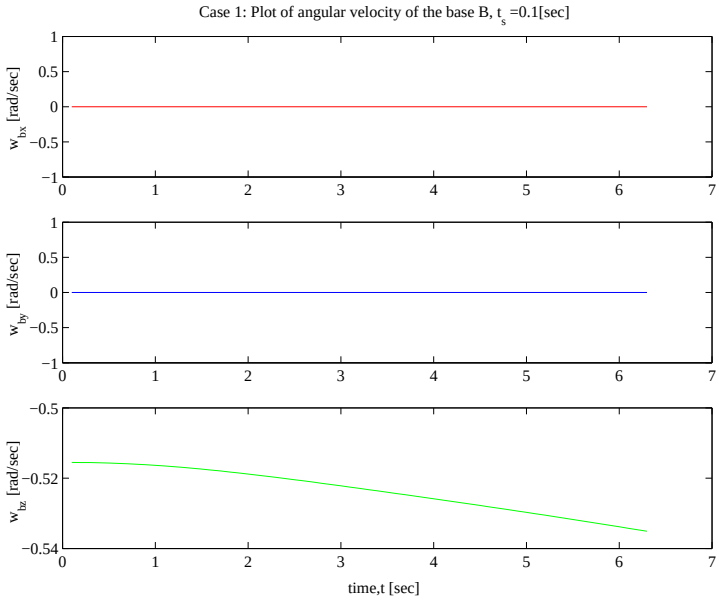

Figure 9.7: Angular velocity of the base, B (Case 1)

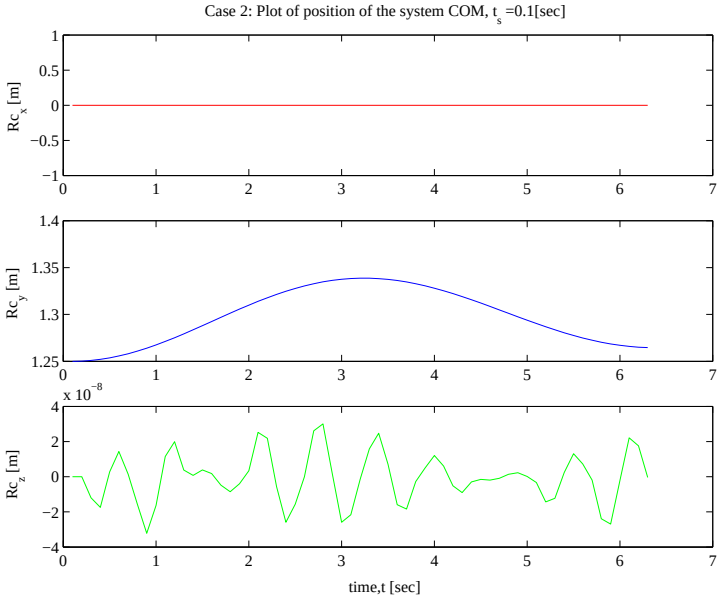

Figure 9.8: System center of mass (Case 2)

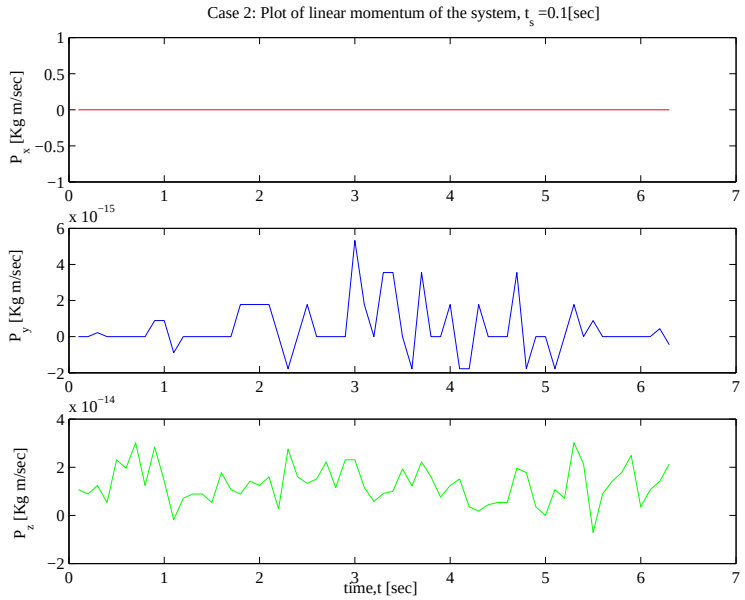

Figure 9.9: Linear momentum of the system (Case 2)

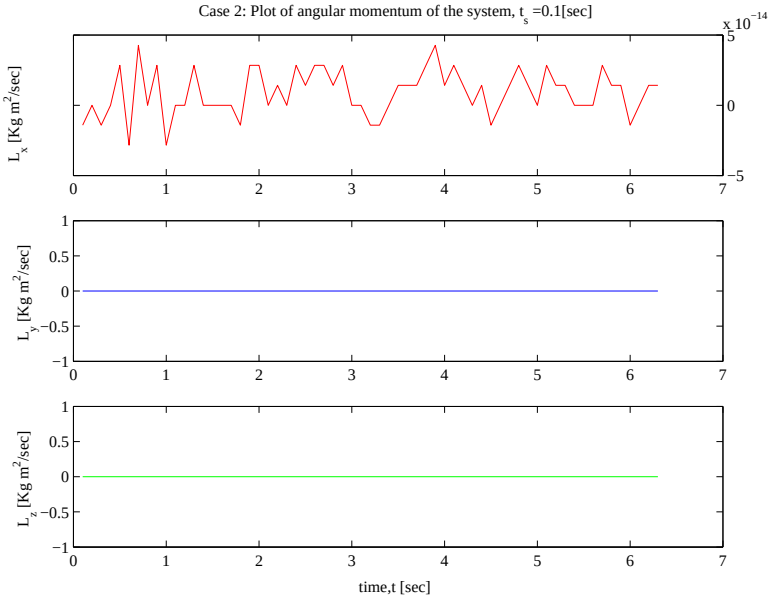

Figure 9.10: Angular momentum of the system (Case 2)

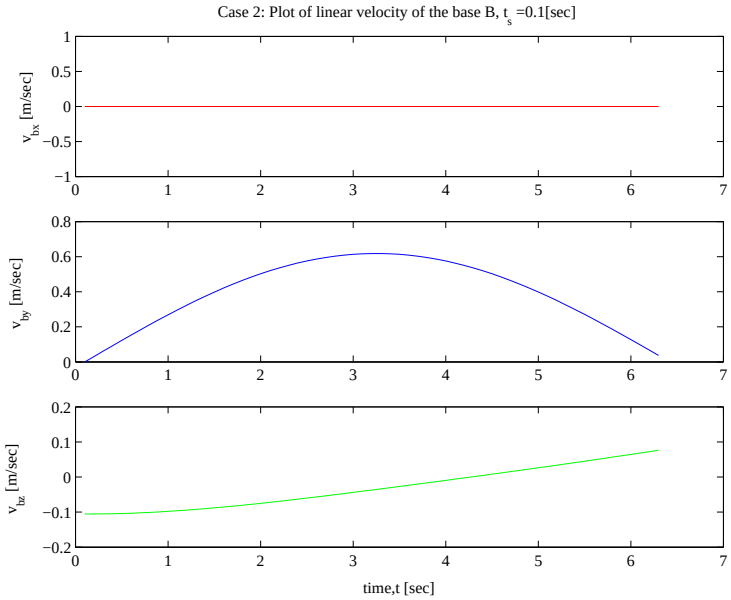

Figure 9.11: Linear velocity of the base, B (Case 2)

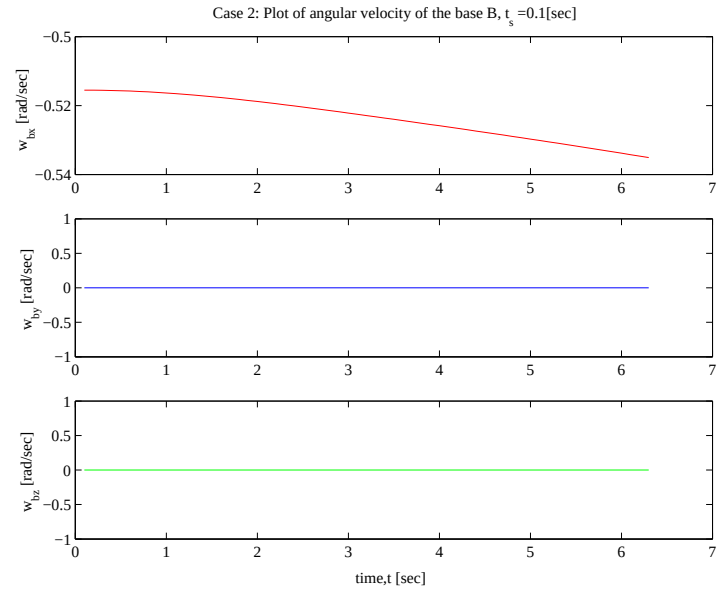

Figure 9.12: Angular velocity of the base, B (Case 2)

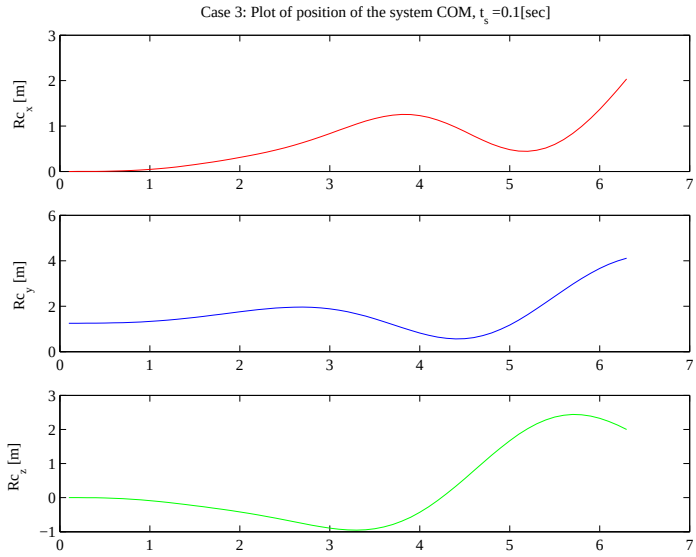

Figure 9.13: System center of mass (Case 3)

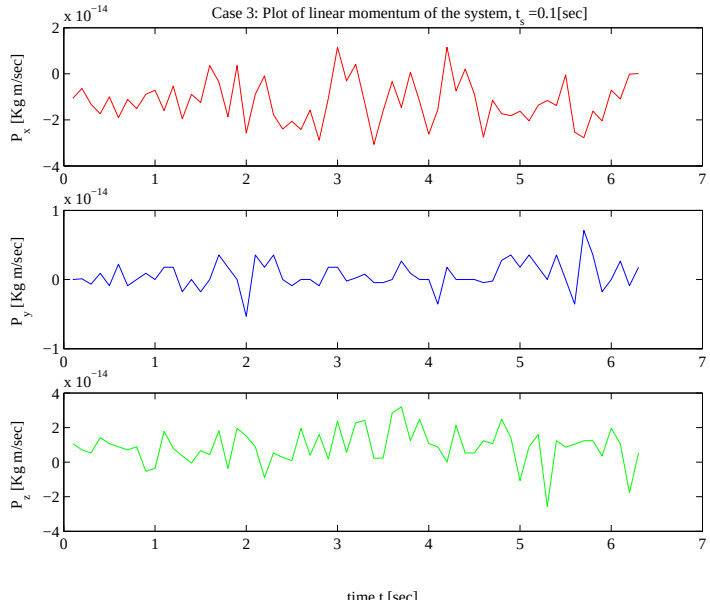

Figure 9.14: Linear momentum of the system (Case 3)

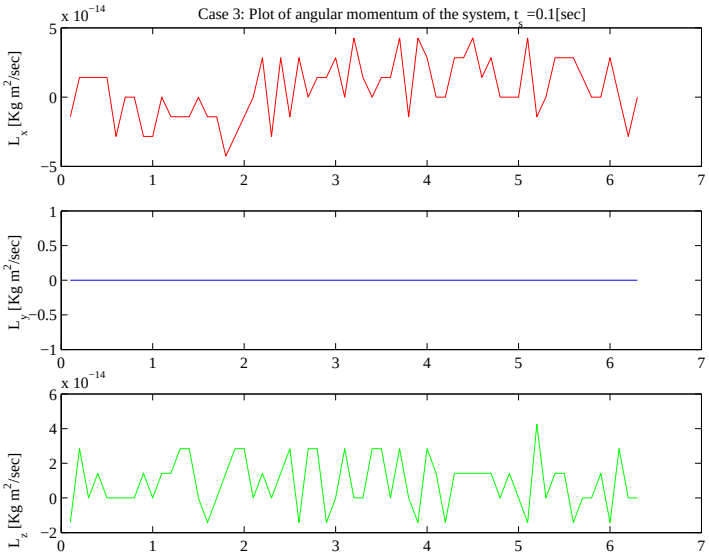

Figure 9.15: Angular momentum of the system (Case 3)

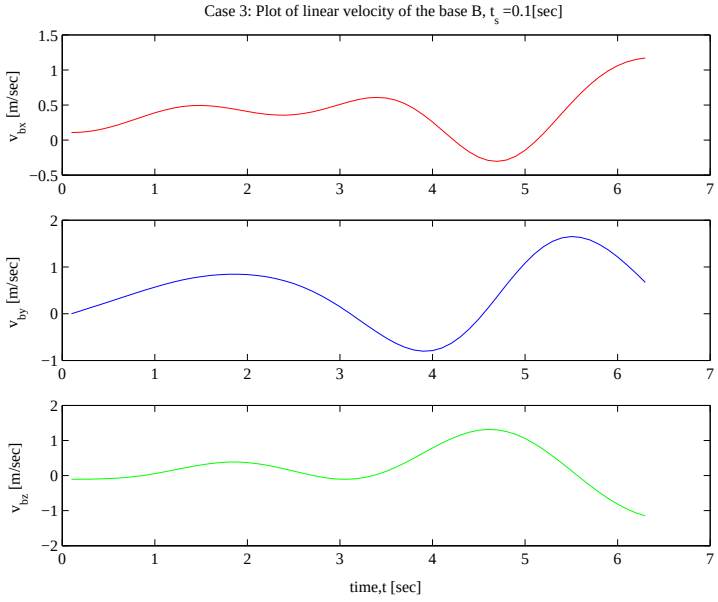

Figure 9.16: Linear velocity of the base, B (Case 3)

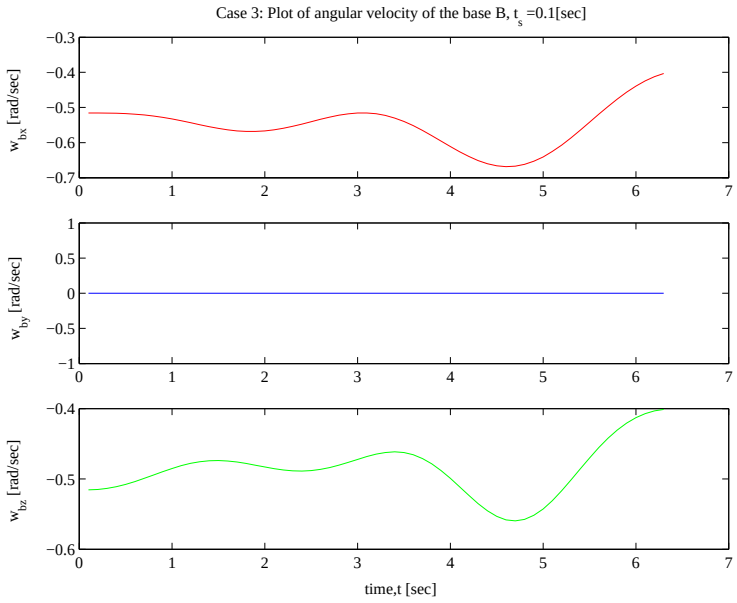

Figure 9.17: Angular velocity of the base, B (Case 3)

Chapter 10

Conclusion and Future work

10.1 Conclusion

Mechanical systems subjected to nonholonomic constraints have received a lot attention in recent years in the field of Geometric Mechanics. However, there is an increasing interest in engineering and robotics in the motion of nonholonomic mechanical systems. Several approaches have been used to describe the dynamics of nonholonomic mechanics such as Hamilton, Lagrange and even Possion method. But we investigate the nonholonomic behaviour of mechanical systems by using Lie Bracket as a tool from Differential Geometry. We classify the mechanical systems into holonomic, nonholonomic and partial nonholonomic by using the repeated Lie Bracket operation to form the accessibility distribution matrix.

We present some examples of nonholonomic mechanical systems where the three categories nonholonomic constraints are imposed on the systems. The examples of the first category are the rolling disk, unicycle and car-like robot where the (kinematics) nonholonomic constraints imposed by the kinematics, such as rolling constraints, which are constraints linear in the velocity.

The kinematically redundant robot is the example of the second category. In the case of kinematic redundancy we will not get a unique solution for the joint velocity vectors for a given end-effectors tasks. On the other hand we will get the infinite number of feasible solutions for the joint velocity vectors for a given end-effectors tasks. For a given end-effectors task or orientation, the minimum-energy solution is the one which guarantees the shortest path in the joint space. In our case we use the minimum-energy solution, i.e., the pseudo-inverse of the jacobian matrix. This inverse kinematics imposed the constraints on the joint velocity vectors to become the corresponding system either holonomic or nonholonomic.

The Free-floating space robot is the example of the third category where the (dynamic) nonholonomic constraints come from the conversation of angular momentum.

In all cases we write the constraints into Pfaffian form, to get the associated kinematics model based on non-linear control theory. After then we investigate the system behaviour on the basis of Lie Bracket operation. Every time we find that the nonholonomic systems are typically controllable in a configuration space of higher dimension than the input space. From a mechanical point of view it is an underactuated mechanism that simple means a

large number of generalized coordinates can be controlled by a few control inputs which can be considered a unique feature of the nonholonomic systems. This will cause a revolutionary change in mechanical design.

we have presented the different simulation results for the Free-floating space robot both in plane and 3D motion. We have shown the linear and angular momentum of the system derived from Pfaffian form as well as system center of mass. We got the expected result both for Free-floating sapce robot in 2D and 3D motion.

10.2 Future work

The future work will be

- To derive the dynamic model of the mechanical systems with kinemtic constraints by using Lagrange-d Alembert equations and to show how it can be partially linearized via feedback.

- To find out an efficient algorithm for nonholonomic motion planning by using higher order Lie Brackets of $\mathbf{g}_j$ to formulate the following extended system:

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q}) u_j + \sum_{i=1+m}^{n} \mathbf{g}_i(\mathbf{q}) u_i$$

  with stabilization criteria.

- Design of nonholonomic manipulator and its control strategy.

- Formulate an algorithm of nonhonomic criteria based on Lie Bracket operation for the general problem of n-link free-floating space manipulator. This algorithm will be implemented within a multibody simulation environment for inverse kinematics calculations.

- Investigate the problem of switching from nonholonomic to holonomic constraints, depending on system properties (e.g. geometric, kinematic and inertia parameters, configuration state variables).

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Appendix A

Symbol

M	Configuration Space
n	Number of generalized co-ordinates
m	Mobility of the system
k	Number of kinematic constants
k_h	Holonomic kinematic constraints
k_{nh}	Nonholonomic kinematic constraints
p	Nonholonomic constraints
$\mathbf{q}$	Generalised co-ordinates
X_q	Tangent Vector
T_qM	Tangent Space
$\mathbf{g}$	Vector field
$\phi_t^{g_l}$	Flow of the vector field
Δ	Distribution
$\mathbf{G'}(\mathbf{q})$	Distribution matrix
$\mathbf{L}$	Angular momentum
$\mathbf{P}$	Linear momentum
$\mathbf{J}$	Moment of inertia
$\mathbf{T}$	Kinetic energy
$\mathbf{U}$	Potential energy
$\mathbf{J}_w^\dagger(\mathbf{q})$	weighted pseudoinverse

Appendix B

Null Space

The null space of a matrix $\mathbf{A}$ is the set of all vectors $\mathbf{v}$ which solve the equation $\mathbf{Ax} = 0$, a linear subspace of the space of all vectors.

Example 1 :

Let $\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 6 & 4 \\ -3 & 9 & 6 \end{bmatrix}$, find out the nullspace of $\mathbf{A}$.

The equation $\mathbf{Ax} = 0$ ends up being just the single equation $x_1 - 3x_2 + 2x_3 = 0$. So x_1 is leading variable and x_2, x_3 are the free variables. We let $x_2 = u_1$ and $x_3 = u_2$ and solve the equation for x_1. We obtain $x_1 = 3u_1 - 2u_2$. Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3u_1 - 2u_2 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} u_2 \tag{B.1}$$

So the basis of nullspace of $\mathbf{A}$ is the set $\{(3, 1, 0), (-2, 0, 1)\}$ The dimension of the null space is 2.

Example 2 : Unicycle

Let $\mathbf{A} = [\sin\theta \quad -\cos\theta \quad 0]$ find out the nullspace of $\mathbf{A}$.
The equation $\mathbf{Ax} = 0$ ends up being just the single equation $x_1 \sin\theta - x_2 \cos\theta + 0x_3 = 0$. So x_1 is leading variable and x_2 are the free variables. We let $x_2 = u_1 \sin\theta$ and $x_3 = u_2$ and solve the equation for x_1. We obtain $x_1 = u_1 \cos\theta + 0u_2$. Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \cos\theta \\ u_1 \sin\theta \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2 \tag{B.2}$$

So the basis of nullspace of $\mathbf{A}$ is the set $\{(\cos\theta, \sin\theta, 0), (0, 0, 1)\}$. The dimension of the null space is 2.

Example 3: Rolling Disk

Here $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -R\cos\varphi \\ 0 & 1 & 0 & -R\sin\varphi \end{bmatrix}$, find out the nullspace of $\mathbf{A}$.

The equation $\mathbf{Ax} = 0$ ends up being the equations $x_1 - x_4 R\cos\varphi = 0$ and $x_2 - x_4 R\sin\varphi = 0$. So x_1, x_2 are leading variables and x_3, x_4 are the free variables. We let $x_3 = u_2$ and $x_4 = u_1$ and solve the equation for x_1 and x_2. We obtain $x_1 = Ru_1\cos\varphi$ and $x_2 = Ru_1\sin\varphi$ Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} Ru_1\cos\varphi \\ Ru_1\sin\varphi \\ u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} R\cos\varphi \\ R\cos\varphi \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 \tag{B.3}$$

So the basis of nullspace of $\mathbf{A}$ is the set $\{(R\cos\varphi, R\sin\varphi, 0, 1),\ (0, 0, 1, 0)\}$. The dimension of the null space is 2.

Example 4: Car-like Robot

Here $\mathbf{A} = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -l\cos\phi & 0 \end{bmatrix}$, find out the nullspace of $\mathbf{A}$.

The equation $\mathbf{Ax} = 0$ ends up being the equations $x_1\sin\theta - x_2\cos\theta = 0$ and $x_1\sin(\theta+\phi) - x_2\cos(\theta+\phi) - x_3 l\cos\phi = 0$. So x_2, x_3 are leading variables and x_1, x_4 are the free variables. We let $x_1 = u_1\cos\theta$ and $x_4 = u_2$ and solve the equation for x_1 and x_2.
We obtain $x_2 = u_1\sin\theta$ and $x_3 = \frac{u_1}{l}\tan\phi$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} u_1\cos\theta \\ u_1\sin\theta \\ \frac{u_1}{l}\tan\phi \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

So the basis of nullspace of $\mathbf{A}$ is the set $\{(\cos\theta, \sin\theta, \frac{1}{l}\tan\phi, 0),\ (0, 0, 0, 1)\}$. The dimension of the null space is 2.

Appendix C

Lie Bracket

Let $\mathbf{g} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$

Lie Bracket:

$$[\mathbf{g}_1, \mathbf{g}_2] = \left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{q}} \mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{q}} \mathbf{g}_2 \right) = \mathbf{g}_3 \tag{C.1}$$

where

$$\frac{\partial \mathbf{g}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \mathbf{a}}{\partial \mathbf{q}_1} & \frac{\partial \mathbf{a}}{\partial \mathbf{q}_2} & \frac{\partial \mathbf{a}}{\partial \mathbf{q}_3} \\ \frac{\partial \mathbf{b}}{\partial \mathbf{q}_1} & \frac{\partial \mathbf{b}}{\partial \mathbf{q}_2} & \frac{\partial \mathbf{b}}{\partial \mathbf{q}_3} \\ \frac{\partial \mathbf{c}}{\partial \mathbf{q}_1} & \frac{\partial \mathbf{c}}{\partial \mathbf{q}_2} & \frac{\partial \mathbf{c}}{\partial \mathbf{q}_3} \end{bmatrix} \tag{C.2}$$

A Lie Bracket takes two n dimensional vectors and returns a new n-vector. But it is not guaranteed that this vector is linearly independent from the creating two others.

Rolling disk:

$$\mathbf{g}_1 = [R\cos\varphi \quad R\sin\varphi \quad 0 \quad 1]^T \tag{C.3}$$

$$\mathbf{g}_2 = [0 \quad 0 \quad 1 \quad 0]^T \tag{C.4}$$

and

$$\mathbf{q} = [x \quad y \quad \varphi \quad \theta]^T \tag{C.5}$$

and

$$\begin{aligned}\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2] &= \left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{q}}\mathbf{g}_2\right) \\ &= \begin{bmatrix} 0 & 0 & -R\cos\varphi & 0 \\ 0 & 0 & -R\sin\varphi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= [R\sin\varphi \quad -R\cos\varphi \quad 0 \quad 0]^T \end{aligned} \tag{C.6}$$

and

$$\begin{aligned}\mathbf{g}_4 = [\mathbf{g}_2, \mathbf{g}_3] &= \left(\frac{\partial \mathbf{g}_3}{\partial \mathbf{q}}\mathbf{g}_2 - \frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\mathbf{g}_3\right) \\ &= \begin{bmatrix} 0 & 0 & R\cos\varphi & 0 \\ 0 & 0 & R\sin\varphi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -R\sin\varphi & 0 \\ 0 & 0 & R\cos\varphi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R\sin\varphi \\ -R\cos\varphi \\ 0 \\ 0 \end{bmatrix} \\ &= [R\cos\varphi \quad R\sin\varphi \quad 0 \quad 0]^T \end{aligned} \tag{C.7}$$

Car-like Robot:

$$\mathbf{g}_1 = \begin{bmatrix} \cos\theta & \sin\theta & \frac{1}{l}\tan\varphi & 0 \end{bmatrix}^T \tag{C.8}$$

$$\mathbf{g}_2 = [0 \quad 0 \quad 0 \quad 1]^T \tag{C.9}$$

$$\mathbf{q} = [x \quad y \quad \theta \quad \varphi]^T \tag{C.10}$$

We need to compute $\mathbf{g}_3$ and $\mathbf{g}_4$.

$$\begin{aligned}\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2] &= \left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{q}}\mathbf{g}_2\right) \\ &= 0 - \begin{bmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \frac{1}{l\cos^2\varphi} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \frac{-1}{l\cos^2\varphi} & 0 \end{bmatrix}^T \end{aligned} \tag{C.11}$$

and

$$\mathbf{g}_4 = [\mathbf{g}_1, \mathbf{g}_3] = \left(\frac{\partial \mathbf{g}_3}{\partial \mathbf{q}} \mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{q}} \mathbf{g}_3 \right)$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial \varphi} \left(\frac{-1}{l \cos^2 \varphi} \right) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l} \tan\varphi \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \left(\frac{1}{l \cos^2 \varphi} \right) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{l \cos^2 \varphi} \\ 0 \end{bmatrix} \tag{C.12}$$

$$= \begin{bmatrix} -\frac{\sin\theta}{l \cos^2 \varphi} & \frac{\cos\theta}{l \cos^2 \varphi} & 0 & 0 \end{bmatrix}^T \tag{C.13}$$

Appendix D

The Frobenius theorem

The Lie Bracket is a only tool needed to determine whether a system is integrable or not integrable.

(Frobenius's Theorem) A **non-singular** distribution is completely integrable if and only if it is involutive.

As we know that the smooth distribution Δ associated with the m smooth vector field $\{\mathbf{g}_1, \ldots, \mathbf{g}_m\}$ is the map that assigns to each point $\mathbf{q} \in \mathbb{R}^n$ linear subspace of its tangent space, i.e.,

$$\Delta(\mathbf{q}) = \text{span}\{\mathbf{g}_1, \ldots, \mathbf{g}_m\} \subset T_q(\mathbb{R}^n) \tag{D.1}$$

If $\dim \Delta(\mathbf{q}) = m$ then Δ is called non-singular or regular distribution. Otherwise Δ is called singular distribution and $\mathbf{q} \in \mathbb{R}^n$ for which the $\dim \Delta(\mathbf{q}) < m$ are called singular point.

The $n \times m$ matrix , $\mathbf{G}(\mathbf{q})$ can be augmented into an $n \times (m+1)$ matrix, $\mathbf{G}'(\mathbf{q})$, by adding $[g_i, g_j]$ as a new column.

If the rank of $\mathbf{G}'(\mathbf{q})$ is $m+1$ for all Lie Brackets then it is immediately known that the system is nonholonomic.

If the rank of $\mathbf{G}'(\mathbf{q})$ is m for all Lie brackets, then the system is completely integrable.

We will see the following examples.

Example 1 : Singular distribution

Let us consider the following system .

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ 1+q_3 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} q_1 q_2 \\ (1+q_3)q_2 \\ q_2 \end{bmatrix} u_2 + \begin{bmatrix} q_1 \\ q_2 \\ 0 \end{bmatrix} u_3 \tag{D.2}$$

$$= \mathbf{g}_1 u_1 + \mathbf{g}_2 u_2 + \mathbf{g}_3 u_3 \tag{D.3}$$

$$= \mathbf{G}(\mathbf{q})\mathbf{u} \tag{D.4}$$

For any $\mathbf{q} \in \mathbb{R}^3$, $\mathbf{g}_2(\mathbf{q}) = q_2\mathbf{g}_1(\mathbf{q})$ which implies that the vector fields are linearly depends over all of $\mathbf{q} \in \mathbb{R}^3$, hence the distribution is singular if , q_1 and $q_2 \neq 0$ because

$$\begin{aligned} &\det\mathbf{G}(\mathbf{q}) = 0 \\ &\text{rank}\mathbf{G}(\mathbf{q}) = 2, i.e. \dim\Delta(\mathbf{q}) = 2 < m(= 3) \end{aligned}$$

If both q_1 and $q_2 = 0$ then $\text{rank}\mathbf{G}(\mathbf{q}) = \Delta(\mathbf{q})$. So $\Delta(\mathbf{q})$ depends on $\mathbf{q}$.

Example 2: Motion trapped on a sphere

We consider the following Pfaffian constraints

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = 0 \tag{D.5}$$

Let $\dot{q}_2 = -q_1u_1$ and $\dot{q}_3 = -q_1u_2$ then we obtain

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} q_2 \\ -q_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} q_3 \\ 0 \\ -q_1 \end{bmatrix} u_2 = \mathbf{g}_1u_1 + \mathbf{g}_2u_2 = \mathbf{G}(\mathbf{q})\mathbf{u} \tag{D.6}$$

then we get

$$\begin{aligned} &\text{rank}\mathbf{G}(\mathbf{q}) = 2 \\ &\dim\Delta = \dim(\text{span}\{\mathbf{g}_1, \mathbf{g}_2\})(\mathbf{q}) = 2(\text{non-singular distribution}) \end{aligned}$$

after Lie Bracket operation, we get

$$\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2]^T = [0 \quad q_3 \quad -q_2]^T$$

Then we get the following accessibility distribution matrix:

$$\mathbf{G}'(\mathbf{q}) = \begin{bmatrix} q_2 & q_3 & 0 \\ -q_1 & 0 & q_3 \\ 0 & -q_1 & -q_2 \end{bmatrix} \tag{D.7}$$

Then we get

$$\begin{aligned} &\det\mathbf{G}'(\mathbf{q}) = 0 \\ &\mathbf{G}'(\mathbf{q}) = 2 \neq (n = 3) \end{aligned}$$

which means that g_3 is never linearly independent of g_1 and g_2. Therefore by Frobenius theorem the system is completely integrable i.e. the system is holonomic .
We can construct the system by integrating the Pfaffian constraint as follows:

$$q_1^2 + q_2^2 + q_3^2 - r^2 = 0$$

It is a sphere of radius $r(t) = r$ and the foliation is generated by selecting any $r \in (0, \infty)$.

Example 3: Unicycle

$$\mathbf{g}_1 = [R\cos\theta \quad R\sin\theta \quad 0]^T \tag{D.8}$$
$$\mathbf{g}_2 = [0\,0\,1]^T \tag{D.9}$$

then we get

$$\text{rank}\mathbf{G}(\mathbf{q}) = 2$$
$$\dim\Delta = \dim(\text{span}\{\mathbf{g}_1, \mathbf{g}_2\})(\mathbf{q}) = 2(\text{non-singular distribution})$$

after Lie Bracket operation, we get

$$\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2]^T = [\sin\theta \quad -\cos\theta \quad 0]^T$$

Then we get the following accessibility distribution matrix:

$$\mathbf{G}'(\mathbf{q}) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{bmatrix} \tag{D.10}$$

Then we get

$$\det\mathbf{G}'(\mathbf{q}) = -R^2 \neq 0$$
$$\text{rank}\mathbf{G}'(\mathbf{q}) = 3 = n$$

which means that $\mathbf{g}_3$ is linearly independent of $\mathbf{g}_1$ and $\mathbf{g}_2$. So by Frobenius theorem the system is not integrable i.e. the system is nonholonomic.

Example 4: The rolling disk

We get the following accessibility distribution matrix:

$$\mathbf{G}'(\mathbf{q}) = \begin{bmatrix} R\cos\varphi & 0 & R\sin\varphi & R\cos\varphi \\ R\sin\varphi & 0 & -R\cos\varphi & R\sin\varphi \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{D.11}$$

Then we get

$$\det\mathbf{G}'(\mathbf{q}) = -R^2 \neq 0$$
$$\text{rank}\mathbf{G}'(\mathbf{q}) = 4 = n$$

which means that $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ and $\mathbf{g}_4$ are linearly independent. So by Frobenius theorem the system is not integrable i.e. the system is nonholonomic.

Example 5 : Car-like Robot

We get the following accessibility distribution matrix:

$$\mathbf{G}'(\mathbf{q}) = \begin{bmatrix} \cos\theta & 0 & 0 & -\frac{-\sin\theta}{l\cos^2\phi} \\ \sin\theta & 0 & 0 & \frac{\cos\theta}{l\cos^2\phi} \\ \frac{1}{l}\tan\phi & 0 & -\frac{1}{l\cos^2\phi} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{D.12}$$

Then we get

$$\begin{aligned} \det\mathbf{G}'(\mathbf{q}) &= \frac{1}{l\cos^2\phi} \neq 0 \\ \text{rank}\mathbf{G}'(\mathbf{q}) &= 4 = n \end{aligned}$$

which means that $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ and $\mathbf{g}_4$ are linearly independent. So by Frobenius theorem the system is not integrable i.e. the system is nonholonomic.

Appendix E

Planar Space Robot

E.1 Two-Body Planar Space Robot

1. First Approach [1]

$$\mathbf{v}_1 = \dot{\mathbf{r}}_{cl} = \begin{bmatrix} \dot{r}_{cl,x} \\ \dot{r}_{cl,y} \end{bmatrix} = \begin{bmatrix} -k_{11}\sin\theta_1 & -k_{12}\sin\theta_2 \\ k_{11}\cos\theta_1 & k_{12}\cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{E.1}$$

$$\begin{aligned} \mathbf{v}_1^2 =& \dot{\mathbf{r}}_{cl}^T \dot{\mathbf{r}}_{cl} \\ =& \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} -k_{11}\sin\theta_1 & k_{11}\cos\theta_1 \\ -k_{12}\sin\theta_2 & k_{12}\cos\theta_2 \end{bmatrix} \begin{bmatrix} -k_{11}\sin\theta_1 & -k_{12}\sin\theta_2 \\ k_{11}\cos\theta_1 & k_{12}\cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ =& \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} k_{11}^2 & k_{11}k_{12}\cos(\theta_2-\theta_1) \\ k_{21}k_{22}\cos(\theta_2-\theta_1) & k_{22}^2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \end{aligned} \tag{E.2}$$

Similarly

$$\begin{aligned} \mathbf{v}_2^2 =& \dot{\mathbf{r}}_{c2}^T \dot{\mathbf{r}}_{c2} \\ =& \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} k_{21}^2 & k_{21}k_{22}\cos(\theta_2-\theta_1) \\ k_{21}k_{22}\cos(\theta_2-\theta_1) & k_{22}^2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \end{aligned} \tag{E.3}$$

The total kinetic energy of the system

$$T = \sum_{i=1}^{2} T_i = \sum \left(\frac{1}{2} m_i \dot{\mathbf{r}}_{cl}^T \dot{\mathbf{r}}_{cl} + \frac{1}{2} J_i \dot{\theta}_1^2 \right), i = 1, 2 \tag{E.4}$$

$$T = T_1 + T_2 = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12}(\theta_1,\theta_2) \\ b_{21}(\theta_1,\theta_2) & b_{22} \end{bmatrix} \tag{E.5}$$

where

$$b_{11} = J_1 + m_1 k_{11}^2 + m_2 k_{21}^2 \tag{E.6}$$
$$b_{12}(\theta_1, \theta_2) = (m_1 k_{11} k_{12} + m_2 k_{21} k_{22}) \cos(\theta_2 - \theta_1) \tag{E.7}$$
$$b_{21}(\theta_1, \theta_2) = b_{12}(\theta_1, \theta_2) \tag{E.8}$$
$$b_{22} = J_2 + m_1 k_{12}^2 + + m_2 k_{22}^2 \tag{E.9}$$

In our case

$$T = \frac{1}{2}\left[(J_1 + m_1 k_{11}^2 + m_2 k_{21}^2)\dot{\theta}_1^2 + (J_2 + m_1 k12^2 + m_2 k22^2)\dot{\theta}_2^2\right] + \frac{1}{2}\left[2b_{12}(\theta_1, \theta_2)\dot{\theta}_1\dot{\theta}_2\right] \tag{E.10}$$
$$= \frac{1}{2}\left(\overline{J}_1\dot{\theta}_1^2 + \overline{J}_2\dot{\theta}_2^2 + 2b_{12}(\theta_1, \theta_2)\dot{\theta}_1\dot{\theta}_2\right) \tag{E.11}$$

where

$$\overline{J}_1 = J_1 + m_1 k_{11}^2 + m_2 k_{21}^2 \tag{E.12}$$
$$\overline{J}_2 = J_2 + m_1 k12^2 + + m_2 k22^2 \tag{E.13}$$

So

$$p_1 = \frac{\partial T}{\partial \dot{\theta}_1} = \overline{J}_1\dot{\theta}_1 + b_{12}(\theta_1, \theta_2)\dot{\theta}_2 \tag{E.14}$$
$$p_2 = \frac{\partial T}{\partial \dot{\theta}_2} = \overline{J}_2\dot{\theta}_2 + b_{12}(\theta_1, \theta_2)\dot{\theta}_1 \tag{E.15}$$
$$\tag{E.16}$$

From the conservation of angular momentum we get the following Pfaffian constraints:

$$\sum_i^2 \mathbf{p}_i = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\overline{J}_1 & b_{12}(\theta_1, \theta_2) \\ b_{12}(\theta_1, \theta_2) & \overline{J}_2\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix} = 0 \tag{E.17}$$

We assume relative angle $\phi_1 = \theta_2 - \theta_1$, so Pfaffian constraints

$$\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\overline{J}_1 & b_{12}(\phi_1) \\ b_{12}(\phi_1) & \overline{J}_2\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\phi}_1 + \dot{\theta}_1\end{bmatrix} = 0 \tag{E.18}$$

or in compact form

$$\begin{bmatrix}\overline{J}_1 + \overline{J}_2 + 2b_{12}(\phi_1) & \overline{J}_2 + b_{12}(\phi_1)\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\phi}_1\end{bmatrix} = 0 \tag{E.19}$$

Let $\mathbf{A}(\mathbf{q}) = \begin{bmatrix}\overline{J}_1 + \overline{J}_2 + 2b_{12}(\phi_1) & \overline{J}_2 + b_{12}(\phi_1)\end{bmatrix}$ and we will find out the null space of $\mathbf{A}(\mathbf{q})$. The equation $\mathbf{A}(\dot{\mathbf{q}}) = 0$ ends up being just the single equation

$$\left(\overline{J}_1 + \overline{J}_2 + 2b_{12}(\phi_1)\right)\dot{\theta}_1 + \left(\overline{J}_2 + b_{12}(\phi_1)\right)\dot{\phi}_1 = 0 \tag{E.20}$$

We let $\dot{\phi}_1 = u, a = \frac{\overline{J}_2}{m_1 k_{11} k_{12} + m_2 k_{21} k_{22}}\overline{J}_2$, and $b = \frac{\overline{J}_1 + \overline{J}_2}{m_1 k_{11} k_{12} + m_2 k_{21} k_{22}}$

$$\dot{\mathbf{q}} = \begin{bmatrix}\dot{\theta}_1 \\ \dot{\phi}_1\end{bmatrix} = \begin{bmatrix}-\frac{a + b\cos\phi_1}{b + 2\cos\phi_1} \\ 1\end{bmatrix} u = \mathbf{G}(\mathbf{q})u \tag{E.21}$$

2. Second Approach (Direct Approach)

Angular momentum of the 1^{st} body

$$L_1 = J_1\dot{\theta}_1 + m_1(k_{11}^2\dot{\theta}_1 + k_{12}^2\dot{\theta}_2 + k_{11}k_{12}\dot{\theta}_2\cos(\theta_2-\theta_1) + k_{11}k_{12}\dot{\theta}_1\cos(\theta_2-\theta_1)) \tag{E.22}$$

Angular momentum of the 2^{nd} body

$$L_2 = J_2\dot{\theta}_2 + m_2(k_{21}^2\dot{\theta}_1 + k_{22}^2\dot{\theta}_2 + k_{21}k_{22}\dot{\theta}_2\cos(\theta_2-\theta_1) + k_{21}k_{22}\dot{\theta}_1\cos(\theta_2-\theta_1)) \tag{E.23}$$

Conservation of angular momentum

$$L_1 + L_2 = 0 \tag{E.24}$$

Let $\phi_1 = \theta_2 - \theta_1$:

$$\begin{bmatrix} J_1\dot{\theta}_1 + m_1(k_{11}^2\dot{\theta}_1 + k_{12}^2\dot{\theta}_2 \\ +k_{11}k_{12}\dot{\theta}_2\cos(\theta_2-\theta_1) + k_{11}k_{12}\dot{\theta}_1\cos(\theta_2-\theta_1)) \\ +J_2(\dot{\phi}_1+\dot{\theta}_1) + m_2(k_{21}^2\dot{\theta}_1 + \\ k_{22}^2(\dot{\phi}_1+\dot{\theta}_1) + (\dot{\phi}_1+\dot{\theta}_1)k_{21}k_{22}\cos\phi_1 + k_{21}k_{22}\dot{\theta}_1\cos\phi_{11}) \end{bmatrix} = 0 \tag{E.25}$$

$$\begin{bmatrix} (J_1 + m_1k_{11}^2 + m_2k_{12}^2 + 2m_1k_{11}k_{12}\cos\phi_1 \\ +J_2 + m_2k_{21}^2 + m_2k_{22}^2 + 2m_2k_{21}k_{22}\cos\phi_1)\dot{\theta}_1 \\ +(J_2 + m_1k_{12}^2 + m_2k_{22}^2 + m_1k_{11}k_{12}\cos\phi_1 \\ +m_2k_{21}k_{22}\cos\phi_1)\dot{\phi}_1 \end{bmatrix} = 0 \tag{E.26}$$

$$\begin{bmatrix} (J_1 + m_1k_{11}^2 + m_2k_{12}^2 + J_2 + m_2k_{21}^2 + m_2k_{22}^2 \\ +2m_2k_{21}k_{22}\cos\phi_1 + 2m_1k_{11}k_{12}\cos\phi_1)\dot{\theta}_1 \\ +(J_2 + m_1k_{12}^2 + m_2k_{22}^2 + \\ m_1k_{11}k_{12}\cos\phi_1 + m_2k_{21}k_{22}\cos\phi_1)\dot{\phi}_1 \end{bmatrix} = 0 \tag{E.27}$$

Let

$$\overline{J}_1 = J_1 + m_1k_{11}^2 + m_2k_{21}^2 \tag{E.28}$$

$$\overline{J}_2 = J_2 + m_1k_{12}^2 + m_2k_{22}^2 \quad \text{and} \tag{E.29}$$

$$b_{12}(\phi_1) = (m_1k_{11}k_{12} + m_2k_{21}k_{22})\cos(\phi) \tag{E.30}$$

then we obtain

$$\left(\overline{J}_1 + \overline{J}_2 + 2b_{12}(\phi_1)\right)\dot{\theta}_1 + \left(\overline{J}_2 + b_{12}(\phi_1)\right)\dot{\phi}_1 = 0 \tag{E.31}$$

or in compact form

$$\begin{bmatrix} \overline{J}_1 + \overline{J}_2 + 2b_{12}(\phi_1) & \overline{J}_2 + b_{12}(\phi_1) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \end{bmatrix} = 0 \tag{E.32}$$

we get the same equation from these two approaches.

E.2 Three-Body Planar Space Robot

From the geometrical constraints of the body we get the following

$$\begin{bmatrix} r_{cl,x} \\ r_{cl,y} \end{bmatrix} = k_{11} \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + k_{12} \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} + k_{13} \begin{bmatrix} \cos\theta_3 \\ \sin\theta_3 \end{bmatrix} \tag{E.33}$$

$$\begin{bmatrix} r_{c2,x} \\ r_{c2,y} \end{bmatrix} = k_{21} \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + k_{22} \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} + k_{23} \begin{bmatrix} \cos\theta_3 \\ \sin\theta_3 \end{bmatrix} \tag{E.34}$$

$$\begin{bmatrix} r_{c3,x} \\ r_{c3,y} \end{bmatrix} = k_{31} \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + k_{32} \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} + k_{33} \begin{bmatrix} \cos\theta_3 \\ \sin\theta_3 \end{bmatrix} \tag{E.35}$$

with the abbreviations

$$k_{11} = \frac{-(m_2+m_3)(l_1-d_1)}{m_t} \qquad k_{12} = -\frac{m_2 d_2 + l_2 m_3}{m_t}$$

$$k_{13} = -\frac{m_3 d_3}{m_t} \qquad k_{21} = \frac{m_1(l_1-d_1)}{m_t}$$

$$k_{22} = \frac{m_1 d_2 - (l_2-d_2)m_3}{m_t} \qquad k_{23} = -\frac{m_3 d_3}{m_t}$$

$$k_{31} = k_{21} \qquad k_{32} = \frac{m_1 l_2 + (l_2-d_2)m_3}{m_t}$$

$$k_{33} = \frac{d_3(m_1+m_2)}{m_t}$$

The velocity of the 1^{st} body at its center of mass

$$\mathbf{v}_1 = \dot{\mathbf{r}}_{c1} = \begin{bmatrix} \dot{r}_{c1,x} \\ \dot{r}_{c1,y} \end{bmatrix} = \begin{bmatrix} -k_{11}\sin\theta_1 & -k_{12}\sin\theta_2 & -k_{13}\sin\theta_3 \\ k_{11}\cos\theta_1 & k_{12}\cos\theta_2 & k_{13}\cos\theta_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{E.36}$$

The velocity of the 2^{nd} body at its center of mass

$$\mathbf{v}_2 = \dot{\mathbf{r}}_{c1} = \begin{bmatrix} \dot{r}_{c2,x} \\ \dot{r}_{c2,y} \end{bmatrix} = \begin{bmatrix} -k_{21}\sin\theta_1 & -k_{22}\sin\theta_2 & -k_{23}\sin\theta_3 \\ k_{21}\cos\theta_1 & k_{22}\cos\theta_2 & k_{23}\cos\theta_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{E.37}$$

The velocity of the 3^{rd} body at its center of mass

$$\mathbf{v}_3 = \dot{\mathbf{r}}_{c3} = \begin{bmatrix} \dot{r}_{c3,x} \\ \dot{r}_{c3,y} \end{bmatrix} = \begin{bmatrix} -k_{31}\sin\theta_1 & -k_{32}\sin\theta_2 & -k_{33}\sin\theta_3 \\ k_{31}\cos\theta_1 & k_{32}\cos\theta_2 & k_{33}\cos\theta_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{E.38}$$

1. First Approach [1]

$$\mathbf{v}_1^2 = \dot{\mathbf{r}}_{cl}^T \dot{\mathbf{r}}_{cl} = \begin{bmatrix}\dot\theta_1 & \dot\theta_2 & \dot\theta_3\end{bmatrix} \begin{bmatrix} -k_{11}\sin\theta_1 & k_{11}\cos\theta_1 \\ -k_{12}\sin\theta_2 & k_{12}\cos\theta_2 \\ -k_{13}\sin\theta_3 & k_{13}\cos\theta_3 \end{bmatrix}$$

$$\begin{bmatrix} -k_{11}\sin\theta_1 & -k_{12}\sin\theta_1 & -k_{13}\sin\theta_1 \\ k_{11}\cos\theta_1 & k_{12}\cos\theta_2 & k_{13}\cos\theta_3 \end{bmatrix} \begin{bmatrix}\dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3\end{bmatrix} \tag{E.39}$$

$$= \begin{bmatrix}\dot\theta_1 & \dot\theta_2 & \dot\theta_3\end{bmatrix} \begin{bmatrix} k_{11}^2 & \mathbf{v}_1(1,2) & \mathbf{v}_1(1,3) \\ \mathbf{v}_1(2,1) & k_{12}^2 & \mathbf{v}_1(2,3) \\ \mathbf{v}_1(3,1) & \mathbf{v}_1(3,2) & k_{13}^2 \end{bmatrix} \begin{bmatrix}\dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3\end{bmatrix} \tag{E.40}$$

where

$$\begin{aligned} \mathbf{v}_1(1,2) = \mathbf{v}_1(2,1) &= k_{11}k_{12}\cos(\theta_2-\theta_1) \\ \mathbf{v}_1(1,3) = \mathbf{v}_1(3,1) &= k_{11}k_{13}\cos(\theta_3-\theta_1) \\ \mathbf{v}_1(2,3) = \mathbf{v}_1(3,2) &= k_{12}k_{13}\cos(\theta_3-\theta_2) \end{aligned}$$

similarly

$$\mathbf{v}_2^2 = \begin{bmatrix}\dot\theta_1 & \dot\theta_2 & \dot\theta_3\end{bmatrix} \begin{bmatrix} k_{21}^2 & \mathbf{v}_2(1,2) & \mathbf{v}_2(1,3) \\ \mathbf{v}_2(2,1) & k_{22}^2 & \mathbf{v}_2(2,3) \\ \mathbf{v}_2(3,1) & \mathbf{v}_2(3,2) & k_{23}^2 \end{bmatrix} \begin{bmatrix}\dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3\end{bmatrix} \tag{E.41}$$

where

$$\begin{aligned} \mathbf{v}_2(1,2) = \mathbf{v}_2(2,1) &= k_{21}k_{22}\cos(\theta_2-\theta_1) \\ \mathbf{v}_2(1,3) = \mathbf{v}_2(3,1) &= k_{21}k_{23}\cos(\theta_3-\theta_1) \\ \mathbf{v}_2(2,3) = \mathbf{v}_2(3,2) &= k_{22}k_{23}\cos(\theta_3-\theta_2) \end{aligned}$$

$$\mathbf{v}_3^2 = \begin{bmatrix}\dot\theta_1 & \dot\theta_2 & \dot\theta_3\end{bmatrix} \begin{bmatrix} k_{31}^2 & \mathbf{v}_3(1,2) & \mathbf{v}_3(1,3) \\ \mathbf{v}_3(2,1) & k_{32}^2 & \mathbf{v}_3(2,3) \\ \mathbf{v}_3(3,1) & \mathbf{v}_3(3,2) & k_{33}^2 \end{bmatrix} \begin{bmatrix}\dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3\end{bmatrix} \tag{E.42}$$

where

$$\begin{aligned} \mathbf{v}_3(1,2) = \mathbf{v}_2(2,1) &= k_{31}k_{32}\cos(\theta_2-\theta_1) \\ \mathbf{v}_3(1,3) = \mathbf{v}_2(3,1) &= k_{31}k_{33}\cos(\theta_3-\theta_1) \\ \mathbf{v}_3(2,3) = \mathbf{v}_2(3,2) &= k_{32}k_{33}\cos(\theta_3-\theta_2) \end{aligned}$$

Kinetic energy of the 1^{st} body:

$$T_1 = \frac{1}{2}m_1\dot{\mathbf{v}}_1^2 + \frac{1}{2}J_i\dot{\theta}_1^2 \tag{E.43}$$

$$= \frac{1}{2}J_i\dot{\theta}_1^2 + \frac{1}{2}m_1\begin{bmatrix}\dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3\end{bmatrix}\begin{bmatrix}k_{11}^2 & \mathbf{v}_1(1,2) & \mathbf{v}_1(1,3)\\ \mathbf{v}_1(2,1) & k_{12}^2 & \mathbf{v}_1(2,3)\\ \mathbf{v}_1(3,1) & \mathbf{v}_1(3,2) & k_{13}^2\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix}$$

$$= \frac{1}{2}m_1\begin{bmatrix}k_{11}^2\dot{\theta}_1\dot{\theta}_1 + k_{11}k_{12}\cos(\theta_2-\theta_1)\dot{\theta}_1\dot{\theta}_2\\ +k_{11}k_{13}\cos(\theta_3-\theta_1)\dot{\theta}_3\dot{\theta}_1 + k_{11}k_{12}\cos(\theta_2-\theta_1)\dot{\theta}_1\dot{\theta}_2\\ +k_{12}k_{12}\dot{\theta}_1\dot{\theta}_2 + k_{11}k_{13}\cos(\theta_3-\theta_2)\dot{\theta}_2\dot{\theta}_3\\ +k_{11}k_{13}\cos(\theta_3-\theta_1)\dot{\theta}_1\dot{\theta}_3 + k_{12}k_{13}\cos(\theta_3-\theta_2)\dot{\theta}_2\dot{\theta}_3\\ +k_{13}\dot{\theta}_3\dot{\theta}_3\end{bmatrix} + \frac{1}{2}J_1\dot{\theta}_1^2$$

$$= \frac{1}{2}m_1\left[\sum_{h=1}^{3}\sum_{j=1}^{3}k_{1h}k_{1j}\cos(\theta_h-\theta_j)\dot{\theta}_j\dot{\theta}_h\right] + \frac{1}{2}J_1\dot{\theta}_1^2 \tag{E.44}$$

Kinetic energy of the 2^{nd} body:

$$T_2 = \frac{1}{2}m_2\begin{bmatrix}k_{21}^2\dot{\theta}_1\dot{\theta}_1 + k_{21}k_{22}\cos(\theta_2-\theta_1)\dot{\theta}_1\dot{\theta}_2\\ +k_{21}k_{23}\cos(\theta_3-\theta_1)\dot{\theta}_3\dot{\theta}_1 + k_{21}k_{22}\cos(\theta_2-\theta_1)\dot{\theta}_1\dot{\theta}_2\\ +k_{22}k_{22}\dot{\theta}_1\dot{\theta}_2 + k_{22}k_{23}\cos(\theta_3-\theta_2)\dot{\theta}_2\dot{\theta}_3+\\ k_{21}k_{23}\cos(\theta_3-\theta_1)\dot{\theta}_1\dot{\theta}_3 + k_{22}k_{23}\cos(\theta_3-\theta_2)\dot{\theta}_2\dot{\theta}_3\\ +k_{23}\dot{\theta}_3\dot{\theta}_3\end{bmatrix} + \frac{1}{2}J_2\dot{\theta}_1^2$$

$$= \frac{1}{2}m_1\left[\sum_{h=1}^{3}\sum_{j=1}^{3}k_{2h}k_{2j}\cos(\theta_h-\theta_j)\dot{\theta}_j\dot{\theta}_h\right] + \frac{1}{2}J_2\dot{\theta}_2^2 \tag{E.45}$$

Kinetic energy of the 3^{rd} body:

$$T_3 = \frac{1}{2}m_1\left[\sum_{h=1}^{3}\sum_{j=1}^{3}k_{3h}k_{3j}\cos(\theta_h-\theta_j)\dot{\theta}_j\dot{\theta}_h\right] + \frac{1}{2}J_3\dot{\theta}_3^2 \tag{E.46}$$

Therefore the total energy:

$$T = \frac{1}{2}\dot{\theta}^T\mathbf{B}(\theta)\dot{\theta}\begin{bmatrix}\dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3\end{bmatrix}$$
$$\begin{bmatrix}b_{11} & c_1\cos(\theta_1-\theta_2) & c_2\cos(\theta_1-\theta_3)\\ c_1\cos(\theta_1-\theta_2) & b_{22} & c_3\cos(\theta_2-\theta_3)\\ c_2\cos(\theta_1-\theta_3) & c_3\cos(\theta_2-\theta_3) & b_{33}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix} \tag{E.47}$$

where

$$
\begin{aligned}
c_1 &= m_1 k_{11} k_{12} + m_2 k_{21} k_{22} + m_3 k_{31} k_{32} \\
c_2 &= m_1 k_{11} k_{13} + m_2 k_{21} k_{23} + m_3 k_{31} k_{33} \\
c_3 &= m_1 k_{12} k_{13} + m_2 k_{22} k_{23} + m_3 k_{32} k_{33} \\
b_{11} &= J_1 + m_1 k_{11}^2 + m_2 k_{21}^2 + m_3 k_{31}^2 \\
b_{22} &= J_2 + m_1 k_{12}^2 + m_2 k_{22}^2 + m_3 k_{32}^2 \\
b_{33} &= J_3 + m_1 k_{13}^2 + m_2 k_{23}^2 + m_3 k_{33}^2
\end{aligned}
$$

The Pfaffian constraints:

$$
\sum_{i=1}^{3} p_i = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
\begin{bmatrix} b_{11} & c_1 \cos(\theta_1 - \theta_2) & c_2 \cos(\theta_1 - \theta_3) \\ c_1 \cos(\theta_1 - \theta_2) & b_{22} & c_3 \cos(\theta_2 - \theta_3) \\ c_2 \cos(\theta_1 - \theta_3) & c_3 \cos(\theta_2 - \theta_3) & b_{33} \end{bmatrix}
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = 0 \tag{E.48}
$$

2. Second Approach (Direct Approach)

Angular momentum of the 1^{st} body, 2^{nd} body and 3^{rd} body respectively

$$\mathbf{L}_1 = \mathbf{J}_1 \omega_1 + m_1 (\mathbf{r}_{c1} \times \mathbf{v}_1) \tag{E.49}$$

$$\mathbf{L}_2 = \mathbf{J}_2 \omega_2 + m_2 (\mathbf{r}_{c2} \times \mathbf{v}_2) \tag{E.50}$$

$$\mathbf{L}_3 = \mathbf{J}_3 \omega_3 + m_3 (\mathbf{r}_{c3} \times \mathbf{v}_3) \tag{E.51}$$

and the tilde operator

$$
\begin{aligned}
\tilde{\mathbf{r}}_{c1} &= \begin{bmatrix} 0 & -r_{c1,z} & -r_{c1,y} \\ r_{c1,z} & 0 & -r_{c1,x} \\ -r_{c1,y} & r_{c1,x} & 0 \end{bmatrix} \\
\tilde{\mathbf{r}}_{c2} &= \begin{bmatrix} 0 & -r_{c2,z} & -r_{c2,y} \\ r_{c2,z} & 0 & -r_{c2,x} \\ -r_{c2,y} & r_{c2,x} & 0 \end{bmatrix} \\
\tilde{\mathbf{r}}_{c3} &= \begin{bmatrix} 0 & -r_{c3,z} & -r_{c3,y} \\ r_{c3,z} & 0 & -r_{c3,x} \\ -r_{c3,y} & r_{c3,x} & 0 \end{bmatrix}
\end{aligned} \tag{E.52}
$$

Now we get the angular momentum of the 1^{st} body

$$\mathbf{L}_1 = \mathbf{J}_1\dot{\theta}_1 + m_1(\tilde{\mathbf{r}}_{c1}.\mathbf{v}_1) \tag{E.53}$$

$$\begin{aligned}
&= \begin{bmatrix} 0 \\ 0 \\ J_1\dot{\theta}_1 \end{bmatrix} + m_1 \begin{bmatrix} 0 & -r_{c1,z} & -r_{c1,y} \\ r_{c1,z} & 0 & -r_{c1,x} \\ -r_{c1,y} & r_{c1,x} & 0 \end{bmatrix} \\
&\qquad \begin{bmatrix} -k_{21}\sin\theta_1\dot{\theta}_1 - k_{12}\sin\theta_2\dot{\theta}_2 - k_{13}\sin\theta_3\dot{\theta}_3 \\ k_{21}\cos\theta_1\dot{\theta}_1 - k_{12}\sin\theta_2\dot{\theta}_2 - k_{13}\sin\theta_3\dot{\theta}_3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ J_1\dot{\theta}_1 \end{bmatrix} + m_1 \begin{bmatrix} 0 & 0 & r_{c1,y} \\ 0 & 0 & -r_{c1,x} \\ -r_{c1,y} & r_{c1,x} & 0 \end{bmatrix} \\
&\qquad \begin{bmatrix} -k_{21}\sin\theta_1\dot{\theta}_1 - k_{12}\sin\theta_2\dot{\theta}_2 - k_{13}\sin\theta_3\dot{\theta}_3 \\ k_{21}\cos\theta_1\dot{\theta}_1 - k_{12}\sin\theta_2\dot{\theta}_2 - k_{13}\sin\theta_3\dot{\theta}_3 \\ 0 \end{bmatrix}
\end{aligned} \tag{E.54}$$

since x and y components of $\mathbf{L}$ are zero, we have only

$$L_1 = J_1\dot{\theta}_1 + m_1 \begin{bmatrix} \left(k_{11}^2 + k_{12}k_{11}\cos(\theta_1-\theta_2) + k_{13}k_{11}\cos(\theta_1-\theta_3)\right)\dot{\theta}_1 \\ +\left(k_{12}^2 + k_{11}k_{12}\cos(\theta_1-\theta_2) + k_{13}k_{12}\cos(\theta_2-\theta_3)\right)\dot{\theta}_2 \\ +\left(k_{13}^2 + k_{11}k_{13}\cos(\theta_1-\theta_2) + k_{13}k_{12}\cos(\theta_2-\theta_3)\right)\dot{\theta}_3 \end{bmatrix} \tag{E.55}$$

Similarly we get the angular momentum of the 2nd and 3rd body respectively.

$$L_2 = J_2\dot{\theta}_2 + m_2 \begin{bmatrix} \left(k_{21}^2 + k_{22}k_{21}\cos(\theta_1-\theta_2) + k_{23}k_{21}\cos(\theta_1-\theta_3)\right)\dot{\theta}_1 \\ +\left(k_{22}^2 + k_{21}k_{22}\cos(\theta_1-\theta_2) + k_{23}k_{22}\cos(\theta_2-\theta_3)\right)\dot{\theta}_2 \\ +\left(k_{23}^2 + k_{21}k_{23}\cos(\theta_1-\theta_2) + k_{23}k_{22}\cos(\theta_2-\theta_3)\right)\dot{\theta}_3 \end{bmatrix} \tag{E.56}$$

$$L_3 = J_3\dot{\theta}_3 + m_3 \begin{bmatrix} \left(k_{31}^2 + k_{32}k_{31}\cos(\theta_1-\theta_2) + k_{33}k_{31}\cos(\theta_1-\theta_3)\right)\dot{\theta}_1 \\ +\left(k_{32}^2 + k_{31}k_{32}\cos(\theta_1-\theta_2) + k_{33}k_{32}\cos(\theta_2-\theta_3)\right)\dot{\theta}_2 \\ +\left(k_{33}^2 + k_{31}k_{33}\cos(\theta_1-\theta_2) + k_{33}k_{32}\cos(\theta_2-\theta_3)\right)\dot{\theta}_3 \end{bmatrix} \tag{E.57}$$

Now by the conservation of angular momentum

$$L_1 + L_2 + L_3 = 0 \tag{E.58}$$

we obtain

$$\begin{aligned}
&\begin{bmatrix} m_1\left(k_{11}^2 + k_{12}k_{11}\cos(\theta_1-\theta_2) + k_{13}k_{11}\cos(\theta_1-\theta_3)\right) \\ +m_2\left(k_{21}^2 + k_{22}k_{21}\cos(\theta_1-\theta_2) + k_{23}k_{21}\cos(\theta_1-\theta_3)\right) \\ +m_3\left(k_{31}^2 + k_{32}k_{31}\cos(\theta_1-\theta_2) + k_{33}k_{31}\cos(\theta_1-\theta_3)\right) \end{bmatrix}\dot{\theta}_1 + \\
&\begin{bmatrix} m_1\left(k_{12}^2 + k_{11}k_{12}\cos(\theta_1-\theta_2) + k_{13}k_{12}\cos(\theta_2-\theta_3)\right) \\ +m_2\left(k_{22}^2 + k_{21}k_{22}\cos(\theta_1-\theta_2) + k_{23}k_{22}\cos(\theta_2-\theta_3)\right) \\ +m_3\left(k_{32}^2 + k_{31}k_{32}\cos(\theta_1-\theta_2) + k_{33}k_{32}\cos(\theta_2-\theta_3)\right) \end{bmatrix}\dot{\theta}_2 + \\
&\begin{bmatrix} m_1\left(k_{13}^2 + k_{11}k_{13}\cos(\theta_1-\theta_2) + k_{13}k_{12}\cos(\theta_2-\theta_3)\right) \\ +m_2\left(k_{23}^2 + k_{21}k_{23}\cos(\theta_1-\theta_2) + k_{23}k_{22}\cos(\theta_2-\theta_3)\right) \\ +m_3\left(k_{33}^2 + k_{31}k_{33}\cos(\theta_1-\theta_2) + k_{33}k_{32}\cos(\theta_2-\theta_3)\right) \end{bmatrix}\dot{\theta}_3 + \\
&\qquad J_1\dot{\theta}_1 + J_2\dot{\theta}_2 + J_3\dot{\theta}_3 = 0
\end{aligned} \tag{E.59}$$

or in compact form

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix} b_{11} & c_1\cos(\theta_1-\theta_2) & c_2\cos(\theta_1-\theta_3) \\ c_1\cos(\theta_1-\theta_2) & b_{22} & c_3\cos(\theta_2-\theta_3) \\ c_2\cos(\theta_1-\theta_3) & c_3\cos(\theta_2-\theta_3) & b_{33}\end{bmatrix}\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3\end{bmatrix} = 0 \tag{E.60}$$

where

$$\begin{aligned} c_1 &= m_1k_{11}k_{12} + m_2k_{21}k_{22} + m_3k_{31}k_{32} \\ c_2 &= m_1k_{11}k_{13} + m_2k_{21}k_{23} + m_3k_{31}k_{33} \\ c_3 &= m_1k_{12}k_{13} + m_2k_{22}k_{23} + m_3k_{32}k_{33} \\ b_{11} &= J_1 + m_1k_{11}^2 + m_2k_{21}^2 + m_3k_{31}^2 \\ b_{22} &= J_2 + m_1k_{12}^2 + m_2k_{22}^2 + m_3k_{32}^2 \\ b_{33} &= J_3 + m_1k_{13}^2 + m_2k_{23}^2 + m_3k_{33}^2 \end{aligned}$$

Hence we came to reach the same equation (Eq. (E.48)) as in the first approach. Then we can convert into the following Pfaffian form.

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\end{bmatrix}\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3\end{bmatrix} = 0 \tag{E.61}$$

where

$$\begin{aligned} a_{11} &= b_{11} + c_1\cos(\theta_1-\theta_2) + c_2\cos(\theta_1-\theta_3) \\ a_{12} &= c_1\cos(\theta_1-\theta_2) + b_{22} + c_3\cos(\theta_2-\theta_3) \\ a_{13} &= c_2\cos(\theta_1-\theta_3) + c_3\cos(\theta_2-\theta_3) + b_{33} \end{aligned}$$

Let

$$\begin{aligned} \phi_1 &= \theta_2 - \theta_1 \\ \phi_2 &= \theta_3 - \theta_2 \end{aligned}$$

So we get finally

$$\begin{aligned} a_{11} &= b_{11} + c_1\cos\phi_1 + c_2\cos(\phi_1+\phi_2) \\ a_{12} &= c_1\cos\phi_1 + b_{22} + c_3\cos\phi_2 \\ a_{13} &= c_2\cos(\phi_1+\phi_2) + c_3\cos\phi_2 + b_{33} \end{aligned}$$

Appendix F

Free-floating space robot

Total linear momentum of the system with respect to I-frame:

$$
\begin{aligned}
{}_I\mathbf{P} &= {}_I\,\mathbf{P}_1 + {}_I\,\mathbf{P}_2 \\
&= m_1\,{}_I\mathbf{v}_{com\,1} + m_2\,{}_I\mathbf{v}_{com\,2} \\
&= m_1\,({}_I\mathbf{v}_b + \omega_b \times \mathbf{R}_{IK1}\,{}_{K1}\mathbf{r}_{B_com\,1}) + m_2({}_I\mathbf{v}_b + ({}_I\omega_b + \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\omega_2) \\
&\quad \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\mathbf{r}_{B_com\,2}) \\
&= m_1\,({}_I\mathbf{v}_b - \mathbf{R}_{IK1}\mathbf{r}_1 \times {}_I\omega_b) \\
&\quad + m_2({}_I\mathbf{v}_b + ({}_I\omega_b + \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\omega_2) \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) \\
&= m_1\,{}_I\mathbf{v}_b - \mathbf{R}_{IK1}\mathbf{r}_1 \times {}_I\omega_b + m_2\,{}_I\mathbf{v}_b \\
&\quad + m_2\,({}_I\mathbf{v}_b + ({}_I\omega_b + \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\omega_2) \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) \\
&= m_1\,{}_I\mathbf{v}_b - m_1(\mathbf{R}_{IK1}\mathbf{r}_1 \times {}_I\omega_b) \\
&\quad + m_2\,{}_I\mathbf{v}_b + m_2({}_I\omega_b \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) + m_2(\mathbf{R}_{IK1}\,{}_{K1}\omega_1 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) \\
&\quad + m_2(\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\omega_2 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) \\
&= (m_1 + m_2)\,{}_I\mathbf{v}_b - m_1(\mathbf{R}_{IK1}\mathbf{r}_1 \times {}_I\omega_b) + m_2({}_I\omega_b \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) \\
&\quad + m_2(\mathbf{R}_{IK1}\,{}_{K1}\omega_1 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) + m_2(\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\omega_2 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) \\
&= (m_1 + m_2)\,{}_I\mathbf{v}_b - m_1\mathbf{R}_{IK1}\tilde{\mathbf{r}}_1 \cdot {}_I\omega_b \\
&\quad - m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2 \cdot {}_I\omega_b + m_2(\mathbf{R}_{IK1}\,{}_{K1}\omega_1 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) \\
&\quad + m_2(\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\omega_2 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) \\
&= (m_1 + m_2)\,{}_I\mathbf{v}_b - m_1\mathbf{R}_{IK1}\tilde{\mathbf{r}}_1 \cdot {}_I\omega_b - m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2 \cdot {}_I\omega_b \\
&\quad + m_2(\mathbf{R}_{IK1}\,{}_{K1}\omega_1 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) + m_2(\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2) + \\
&\quad m_2(\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\omega_2 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2)
\end{aligned}
$$

$$
\begin{aligned}
&= (m_1 + m_2)\,{}_I\mathbf{v}_b - (m_1\mathbf{R}_{IK1}\tilde{\mathbf{r}}_1 + m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2)\,{}_I\omega_b \\
&\quad - m_2(\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2 \times \mathbf{R}_{IK1}\tilde{\mathbf{r}}_2 \times \mathbf{R}_{IK1}\,{}_{K1}\omega_1) \\
&\quad - m_2(\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\mathbf{r}_2 \times \mathbf{R}_{IK1}\mathbf{R}_{K1K2}\,{}_{K2}\omega_2) \\
&= (m_1 + m_2)\,{}_I\mathbf{v}_b - (m_1\mathbf{R}_{IK1}\tilde{\mathbf{r}}_1 + m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2)\,{}_I\omega_b \\
&\quad - m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2\mathbf{R}_{IK1}\begin{bmatrix}0\\0\\\dot{q}_1\end{bmatrix} \\
&\quad - m_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\tilde{\mathbf{r}}_2\mathbf{R}_{IK1}\mathbf{R}_{K1K2}\begin{bmatrix}\dot{q}_2\\0\\0\end{bmatrix}
\end{aligned}
$$

where

$$
\tilde{\mathbf{r}}_1 = \begin{bmatrix} 0 & -r_{1z} & r_{1y} \\ r_{1z} & 0 & r_{1x} \\ -r_{1y} & r_{1x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & d_1 \\ 0 & 0 & 0 \\ -d_1 & 0 & 0 \end{bmatrix}
$$

and

$$
\tilde{\mathbf{r}}_2 = \begin{bmatrix} 0 & -r_{2z} & r_{2y} \\ r_{2z} & 0 & r_{2x} \\ -r_{2y} & r_{2x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & d_2 \\ 0 & 0 & 0 \\ -d_2 & 0 & 0 \end{bmatrix}
$$

The angular momentum of the 1^{st} body with respect to K1-frame:

$$
\begin{aligned}
{}_{K1}\mathbf{L}_1 &= {}_K1\mathbf{J}_{COM1}\,{}_{K1}\omega_b + (\,{}_{K1}\mathbf{r}_{B_com\,1} + \mathbf{r}_{B_0}) \times (m_1\,{}_{K1}\mathbf{v}_{com\,1}) \\
&= \mathbf{J}_1\omega_b + (\mathbf{r}_1 + \mathbf{r}_{B_0}) \times (m_1\mathbf{v}_{com1}) \\
&= \mathbf{J}_1\omega_b + (\mathbf{r}_1 \times \mathbf{v}_{com1} + \mathbf{r} \times m_1\mathbf{v}_{com1} \\
&= \mathbf{J}_1\omega_b + m_1\mathbf{r}_1 \times \mathbf{v}_{com1} + \mathbf{r} \times {}_{K1}\mathbf{P}_1 \\
&= \mathbf{J}_1\omega_b + m_1\mathbf{r}_1 \times \mathbf{v}_{com1} + \tilde{\mathbf{r}} \cdot {}_{K1}\mathbf{P}_1 \\
&= \mathbf{J}_1\omega_b + m_1\mathbf{r}_1 \times (\mathbf{v}_b + \omega_b \times \mathbf{r}_1) + \tilde{\mathbf{r}} \cdot {}_{K1}\mathbf{P}_1 \\
&= \mathbf{J}_1\omega_b + m_1(\mathbf{r}_1 \times \mathbf{v}_b + \mathbf{r}_1 \times \omega_b \times \mathbf{r}_1) + \tilde{\mathbf{r}} \cdot {}_{K1}\mathbf{P}_1 \\
&= \mathbf{J}_1\omega_b + m_1(\tilde{\mathbf{r}}_1 \cdot \mathbf{v}_b + \tilde{\mathbf{r}}_1 \cdot \omega_b \times \mathbf{r}_1) + \tilde{\mathbf{r}} \cdot {}_{K1}\mathbf{P}_1 \\
&= \mathbf{J}_1\omega_b + m_1(\tilde{\mathbf{r}}_1 \cdot \mathbf{v}_b - \tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_1\omega_b) + \tilde{\mathbf{r}} \cdot {}_{K1}\mathbf{P}_1 \\
&= m_1\tilde{\mathbf{r}}_1\mathbf{v}_b + (\mathbf{J}_1 - m_1\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_1)\omega_b + \tilde{\mathbf{r}} \cdot {}_{K1}\mathbf{P}_1
\end{aligned}
$$

where

$$
\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_1 = \begin{bmatrix} r_{1y}^2 + r_{1z}^2 & -r_{1x}r_{1y} & -r_{1x}r_{1z} \\ -r_{1x}r_{1y} & r_{1x}^2 + r_{1z}^2 & -r_{1y}r_{1z} \\ -r_{1x}r_{1z} & -r_{1y}r_{1z} & r_{1y}^2 + r_{1x}^2 \end{bmatrix} = - \begin{bmatrix} d_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_1^2 \end{bmatrix}
$$

The angular momentum of the 2^{nd} body with respect to K2-frame:

$$\begin{aligned}
{}_{K2}\mathbf{L}_2 &= {}_{K2}\mathbf{J}_{COM2}\,{}_{K2}\omega_2^{abs} + ({}_{K2}\mathbf{r}_{B_com\,2} + \mathbf{r}_{B_0}) \times (m_2\,{}_{K2}\mathbf{v}_{com\,2}) \\
&= \mathbf{J}_2(\omega_b + \omega_1 + \omega_2) + m_2\mathbf{r}_2 \times \mathbf{v}_{com2} + m_2\mathbf{r}_{B_0} \times \mathbf{v}_{com2} \\
&= \mathbf{J}_2(\omega_b + \omega_1 + \omega_2) + m_2\mathbf{r}_2 \times (\mathbf{v}_b + \omega_2^{abs} \times \mathbf{r}_2) + \mathbf{r} \times m_2\mathbf{v}_{com2} \\
&= \mathbf{J}_2(\omega_b + \omega_1 + \omega_2) + m_2\mathbf{r}_2 \times \mathbf{v}_b + m_2\mathbf{r}_2 \times \omega_2^{abs} \times \mathbf{r}_2 + \mathbf{r} \times {}_{K2}\mathbf{P}_2 \\
&= \mathbf{J}_2(\omega_b + \omega_1 + \omega_2) \\
&\quad + m_2\mathbf{r}_2 \times \mathbf{v}_b + m_2\mathbf{r}_2 \times (\omega_b + \omega_1 + \omega_2) \times \mathbf{r}_2 + \tilde{\mathbf{r}} \cdot {}_{K2}\mathbf{P}_2 \\
&= \mathbf{J}_2(\omega_b + \omega_1 + \omega_2) \\
&\quad + m_2\tilde{\mathbf{r}_2} \cdot \mathbf{v}_b + (\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2)\omega_b + (\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2)\omega_1 \\
&\quad\quad + (\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2)\omega_2 + \tilde{\mathbf{r}} \cdot {}_{K2}\mathbf{P}_2
\end{aligned}$$

where

$$\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2 = \begin{bmatrix} r_{2y}^2 + r_{2z}^2 & -r_{2x}r_{2y} & -r_{2x}r_{2z} \\ -r_{2x}r_{2y} & r_{2x}^2 + r_{2z}^2 & -r_{2y}r_{2z} \\ -r_{2x}r_{2z} & -r_{2y}r_{2z} & r_{2y}^2 + r_{2x}^2 \end{bmatrix} = - \begin{bmatrix} d_2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_2^2 \end{bmatrix}$$

Total angular momentum with respect to I-frame:

$$\begin{aligned}
{}_{K2}\mathbf{L}_2 &= \mathbf{R}_{IK1K1}\mathbf{L}_1 + \mathbf{R}_{IK1}\mathbf{R}_{K1K2K2}\mathbf{L}_2 \\
&= (m_1\tilde{\mathbf{r}}_1 + m_2\tilde{\mathbf{r}}_2)_I\mathbf{v}_b + ({}_{K1}\mathbf{J}_1 - m_1\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_1 + {}_{K2}\,\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2)_I\omega_b \\
&+ ({}_{K2}\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2)_I\omega_1 + ({}_{K2}\mathbf{J}_2 - m_2\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_2)_I\omega_2 + \tilde{\mathbf{r}} \cdot {}_I\,\mathbf{P}
\end{aligned}$$

Pfaffian form:

$$\begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{b}_{21} & \mathbf{b}_{22} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathbf{v}_b \\ \omega_b \end{bmatrix} \\ \\ \begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix} \end{bmatrix} = 0$$

or in compact form

$$\mathbf{A}_{2\times4}(\mathbf{q})\dot{\mathbf{q}}_{2\times4} = 0 \tag{F.1}$$

Let $\mathbf{u}_1 = \dot{\mathbf{q}}_1$ and $\mathbf{u}_2 = \dot{\mathbf{q}}_2$ we get the following:

$$\begin{bmatrix} \begin{bmatrix} \mathbf{v}_b \\ \omega_b \end{bmatrix} \\ \\ \begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{a}_{12}\mathbf{b}_{21} - \mathbf{b}_{11}\mathbf{a}_{22}}{\mathbf{a}_{11}\mathbf{b}_{22} - \mathbf{b}_{21}\mathbf{a}_{12}} & \frac{\mathbf{a}_{12}\mathbf{b}_{22} - \mathbf{b}_{12}\mathbf{a}_{22}}{\mathbf{a}_{11}\mathbf{a}_{22} - \mathbf{a}_{21}\mathbf{a}_{12}} \\ -\frac{-\mathbf{a}_{21}\mathbf{b}_{11} + \mathbf{a}_{11}\mathbf{b}_{21}}{\mathbf{a}_{11}\mathbf{a}_{22} - \mathbf{a}_{21}\mathbf{a}_{12}} & -\frac{\mathbf{a}_{11}\mathbf{b}_{22} - \mathbf{a}_{21}\mathbf{b}_{12}}{\mathbf{a}_{11}\mathbf{a}_{22} - \mathbf{a}_{21}\mathbf{a}_{12}} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} \\ \mathbf{k}_{21} & \mathbf{k}_{22} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

where

$$\mathbf{k}_{11} = \begin{bmatrix} k_{11}^{11} & k_{11}^{12} & k_{11}^{13} \\ k_{11}^{21} & k_{11}^{22} & k_{11}^{23} \\ k_{11}^{31} & k_{11}^{32} & k_{11}^{33} \end{bmatrix} \text{ and } \mathbf{k}_{12} = \begin{bmatrix} k_{12}^{11} & k_{12}^{12} & k_{12}^{13} \\ k_{12}^{21} & k_{12}^{22} & k_{12}^{23} \\ k_{12}^{31} & k_{12}^{32} & k_{12}^{33} \end{bmatrix}$$

$$\mathbf{k}_{21} = \begin{bmatrix} k_{21}^{11} & k_{21}^{12} & k_{21}^{13} \\ k_{21}^{21} & k_{21}^{22} & k_{21}^{23} \\ k_{21}^{31} & k_{21}^{32} & k_{21}^{33} \end{bmatrix} \text{ and } \mathbf{k}_{22} = \begin{bmatrix} k_{22}^{11} & k_{22}^{12} & k_{22}^{13} \\ k_{22}^{21} & k_{22}^{22} & k_{22}^{23} \\ k_{22}^{31} & k_{22}^{32} & k_{22}^{33} \end{bmatrix}$$

Then we can get as in compact form:

$$\dot{\mathbf{q}}_{4\times1} = \mathbf{G}_{4\times2}(\mathbf{q})\mathbf{A}_{2\times1} \tag{F.2}$$

After expanding we get

$$\mathbf{G}_{12\times6}(\mathbf{q}) = \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} \\ \mathbf{k}_{21} & \mathbf{k}_{22} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} k_{11}^{11} & k_{11}^{12} & k_{11}^{13} \\ k_{11}^{21} & k_{11}^{22} & k_{11}^{23} \\ k_{11}^{31} & k_{11}^{32} & k_{11}^{33} \end{bmatrix} & \begin{bmatrix} k_{12}^{11} & k_{12}^{12} & k_{12}^{13} \\ k_{12}^{21} & k_{12}^{22} & k_{12}^{23} \\ k_{12}^{31} & k_{12}^{32} & k_{12}^{33} \end{bmatrix} \\ \begin{bmatrix} k_{21}^{11} & k_{21}^{12} & k_{21}^{13} \\ k_{21}^{21} & k_{21}^{22} & k_{21}^{23} \\ k_{21}^{31} & k_{21}^{32} & k_{21}^{33} \end{bmatrix} & \begin{bmatrix} k_{22}^{11} & k_{22}^{12} & k_{22}^{13} \\ k_{22}^{21} & k_{22}^{22} & k_{22}^{23} \\ k_{22}^{31} & k_{22}^{32} & k_{22}^{33} \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{bmatrix} \tag{F.3}$$

Now we reduce this matrix for our particular case. Then 7th, 8th, 11th and 12th rows will be deleted from $\mathbf{G}_{12\times6}(\mathbf{q})$ where

$$\mathbf{u} = (0 \quad 0 \quad \dot{\mathbf{q}}_1 \quad \dot{\mathbf{q}}_2 \quad 0 \quad 0)^T$$

So we obtain our kinematic model

$$\mathbf{G}'_{8\times2}(\mathbf{q}) = \text{span}\{\mathbf{g}'_3, \mathbf{g}'_4\} \tag{F.4}$$

$$\dot{\mathbf{q}}_{8\times1} = \mathbf{G}'_{8\times2}(\mathbf{q})\mathbf{u}_{2\times1} \tag{F.5}$$

where

$$\mathbf{g}'_3 = (k_{11}^{13} \quad k_{11}^{23} \quad k_{11}^{33} \quad k_{21}^{13} \quad k_{21}^{23} \quad 1 \quad 0)^T \tag{F.6}$$

$$\mathbf{g}'_4 = (k_{12}^{11} \quad k_{12}^{21} \quad k_{12}^{31} \quad k_{22}^{11} \quad k_{22}^{21} \quad k_{22}^{31} \quad 0 \quad 1)^T \tag{F.7}$$

Lie Bracket

We have already got $\mathbf{g}'_1, \mathbf{g}'_2, \mathbf{g}'_3, \mathbf{g}'_4, \mathbf{g}'_5, \mathbf{g}'_6$ so we would like to find out $\mathbf{g}'_7, \mathbf{g}'_8$. But we have not calculated $\mathbf{g}'_7, \mathbf{g}'_8$ because it is too heavy. Maple calculation did not return any solution.

Appendix G

Calculation with Maple Software

G.1 Redundant Manipulator

G.1.1 One Link Manipulator

>x := q1+L*cos(q3);

$$x := q1 + L\cos(q3)$$

>y := q2+L*sin(q3);

$$y := q2 + L\sin(q3)$$

>Jq:=<<diff(x,q1),diff(y,q1)>|<diff(x,q2),diff(y,q2)>|<diff(x,q3),diff (y,q3)>>;

$$Jq := \begin{bmatrix} 1 & 0 & -L\sin(q3) \\ 0 & 1 & L\cos(q3) \end{bmatrix}$$

>W:=<<1,0,0>|<0,1,0>|<0,0,w^2 >>;

$$\mathrm{W} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w^2 \end{bmatrix}$$

>inW:=Adjoint(W)/Determinant(W);

$$inW := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{w^2} \end{bmatrix}$$

>AA:=Jq.inW.Transpose(Jq);

$$AA := \begin{bmatrix} 1 + \frac{L^2\sin(q3)^2}{w^2} & -\frac{L^2\sin(q3)\cos(q3)}{w^2} \\ -\frac{L^2\sin(q3)\cos(q3)}{w^2} & 1 + \frac{L^2\cos(q3)^2}{w^2} \end{bmatrix}$$

>inAA:=Adjoint(AA)/Determinant(AA);

$$\text{inAA} := \begin{bmatrix} \dfrac{w^2\left(1+\dfrac{L^2\cos(q3)^2}{w^2}\right)}{\%1} & \dfrac{L^2\sin(q3)\cos(q3)}{\%1} \\ \dfrac{L^2\sin(q3)\cos(q3)}{\%1} & \dfrac{w^2\left(1+\dfrac{L^2\sin(q3)^2}{w^2}\right)}{\%1} \end{bmatrix}$$

$$\%1 := w^2 + L^2\cos(q3)^2 + L^2\sin(q3)^2$$

$> pseJq := inW.Transpose(Jq).inAA;$

$$\text{pseJq} := \left[\frac{w^2\left(1+\dfrac{L^2\cos(q3)^2}{w^2}\right)}{\%1}, \frac{L^2\sin(q3)\cos(q3)}{\%1}\right]$$

$$\left[\frac{L^2\sin(q3)\cos(q3)}{\%1}, \frac{w^2\left(1+\dfrac{L^2\sin(q3)^2}{w^2}\right)}{\%1}\right]$$

$$\left[-\frac{L\sin(q3)\left(1+\dfrac{L^2\cos(q3)^2}{w^2}\right)}{\%1} + \frac{L^3\cos(q3)^2\sin(q3)}{w^2\,\%1},\right.$$

$$\left. -\frac{L^3\sin(q3)^2\cos(q3)}{w^2\,\%1} + \frac{L\cos(q3)\left(1+\dfrac{L^2\sin(q3)^2}{w^2}\right)}{\%1}\right]$$

$$\%1 := w^2 + L^2\cos(q3)^2 + L^2\sin(q3)^2$$

$> Gq := simplify(pseJq);$

$$Gq := \begin{bmatrix} \dfrac{w^2+L^2\cos(q3)^2}{w^2+L^2} & \dfrac{L^2\sin(q3)\cos(q3)}{w^2+L^2} \\ \dfrac{L^2\sin(q3)\cos(q3)}{w^2+L^2} & \dfrac{w^2+L^2-L^2\cos(q3)^2}{w^2+L^2} \\ -\dfrac{L\sin(q3)}{w^2+L^2} & \dfrac{L\cos(q3)}{w^2+L^2} \end{bmatrix}$$

>g1:=<Gq[1,1],Gq[2,1],Gq[3,1]>;

$$g1 := \begin{bmatrix} \dfrac{w^2+L^2\cos(q3)^2}{w^2+L^2} \\ \dfrac{L^2\sin(q3)\cos(q3)}{w^2+L^2} \\ -\dfrac{L\sin(q3)}{w^2+L^2} \end{bmatrix}$$

>g2:=<Gq[1,2],Gq[2,2],Gq[3,2]>;

$$g2 := \begin{bmatrix} \dfrac{L^2 \sin(q3)\cos(q3)}{w^2+L^2} \\ \dfrac{w^2+L^2-L^2\cos(q3)^2}{w^2+L^2} \\ \dfrac{L\cos(q3)}{w^2+L^2} \end{bmatrix}$$

>dffg1:=<<diff(g1[1],q1),diff(g1[2],q1),diff(g1[3],q1)>|
<diff(g1[1],q2),diff(g1[2],q2),diff(g1[3],q2)>|
<diff(g1[1],q3),diff(g1[2],q3),diff(g 1[3],q3)>>;

$$dffg1 := \begin{bmatrix} 0 & 0 & -\dfrac{2\,L^2\sin(q3)\cos(q3)}{w^2+L^2} \\ 0 & 0 & \dfrac{L^2\cos(q3)^2}{w^2+L^2} - \dfrac{L^2\sin(q3)^2}{w^2+L^2} \\ 0 & 0 & -\dfrac{L\cos(q3)}{w^2+L^2} \end{bmatrix}$$

>dffg2:=<<diff(g2[1],q1),diff(g2[2],q1),diff(g2[3],q1)>|
<diff(g2[1],q2),diff(g2[2],q2),diff(g2[3],q2)>|
<diff(g2[1],q3),diff(g2[2],q3),diff(g 2[3],q3)>>;

$$dffg2 := \begin{bmatrix} 0 & 0 & \dfrac{L^2\cos(q3)^2}{w^2+L^2} - \dfrac{L^2\sin(q3)^2}{w^2+L^2} \\ 0 & 0 & \dfrac{2\,L^2\sin(q3)\cos(q3)}{w^2+L^2} \\ 0 & 0 & -\dfrac{L\sin(q3)}{w^2+L^2} \end{bmatrix}$$

>g3:=dffg2.g1-dffg1.g2;
>simplify(g3);

$$\begin{bmatrix} \dfrac{\sin(q3)\,L^3}{(w^2+L^2)^2} \\ -\dfrac{L^3\cos(q3)}{(w^2+L^2)^2} \\ \dfrac{L^2}{(w^2+L^2)^2} \end{bmatrix}$$

>Gq_{accs} := << g1 > | < g2 > | < g3 >>;
Gq_accs :=
$$\left[\frac{w^2+L^2\cos(q3)^2}{w^2+L^2}, \frac{L^2\sin(q3)\cos(q3)}{w^2+L^2},\right.$$

$$-\frac{(\frac{L^2\cos(q3)^2}{w^2+L^2}-\frac{L^2\sin(q3)^2}{w^2+L^2})\,L\sin(q3)}{w^2+L^2}+\frac{2\,L^3\sin(q3)\cos(q3)^2}{(w^2+L^2)^2}\Bigg]$$

$$\left[\frac{L^2\sin(q3)\cos(q3)}{w^2+L^2},\ \frac{w^2+L^2-L^2\cos(q3)^2}{w^2+L^2},\right.$$

$$-\frac{2\,L^3\sin(q3)^2\cos(q3)}{(w^2+L^2)^2}-\frac{(\frac{L^2\cos(q3)^2}{w^2+L^2}-\frac{L^2\sin(q3)^2}{w^2+L^2})\,L\cos(q3)}{w^2+L^2}\Bigg]$$

$$\left[-\frac{L\sin(q3)}{w^2+L^2},\ \frac{L\cos(q3)}{w^2+L^2},\ \frac{L^2\sin(q3)^2}{(w^2+L^2)^2}+\frac{L^2\cos(q3)^2}{(w^2+L^2)^2}\right]$$

> *Rank*(Gq_accs);

3

G.1.2 Two Link Manipulator

>zeta1 := q1+l1*cos(q3)+l2*cos(q4)

$$\zeta_1 := q1 + l1\ \cos(q3) + l2\ \cos(q4)$$

>zeta2:=q2+l1*sin(q3)+l2*sin(q4);

$$\zeta_2 := q2 + l1\ \sin(q3) + l2\ \sin(q4)$$

>Jq:=<<diff(zeta1,q1),diff(zeta2,q1)>|<diff(zeta1,q2),diff(zeta2,q2)>|
<diff(zeta1,q3),diff(zeta2,q3)>|<diff(zeta1,q4),diff(zeta2,q4)>>;

$$Jq := \begin{bmatrix} 1 & 0 & -l1\ \sin(q3) & -l2\ \sin(q4) \\ 0 & 1 & l1\ \cos(q3) & l2\ \cos(q4) \end{bmatrix}$$

>W:=<<1,0,0,0>|<0,1,0,0>|<0,0,1,0>|<0,0,0,w^2>>;

$$W := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & w^2 \end{bmatrix}$$

>inW:=Adjoint(W)/Determinant(W);
>AA:=Jq.inW.Transpose(Jq);
>inAA:=Adjoint(AA)/Determinant(AA);
>pseJq:=inW.Transpose(Jq).inAA;
>Gq:=simplify(pseJq);
>Rank(Gq);

2

>g1:=<Gq[1,1],Gq[2,1],Gq[3,1],Gq[4,1]>;

$g1 :=$

$$\begin{bmatrix} \frac{w^2+l1^2(\cos(q3))^2w^2+l2^2(\cos(q4))^2}{w^2+l1^2w^2+l1^2l2^2(\cos(q4))^2-2\,l1^2l2^2(\cos(q4))^2(\cos(q3))^2+l2^2+l2^2l1^2(\cos(q3))^2-2\,l1^2\sin(q3)\cos(q3)l2^2\sin(q4)\cos(q4)} \\ \frac{l1^2\sin(q3)\cos(q3)w^2+l2^2\sin(q4)\cos(q4)}{w^2+l1^2w^2+l1^2l2^2(\cos(q4))^2-2\,l1^2l2^2(\cos(q4))^2(\cos(q3))^2+l2^2+l2^2l1^2(\cos(q3))^2-2\,l1^2\sin(q3)\cos(q3)l2^2\sin(q4)\cos(q4)} \\ -\frac{l1\left(\sin(q3)w^2+\sin(q3)l2^2(\cos(q4))^2-\cos(q3)l2^2\sin(q4)\cos(q4)\right)}{w^2+l1^2w^2+l1^2l2^2(\cos(q4))^2-2\,l1^2l2^2(\cos(q4))^2(\cos(q3))^2+l2^2+l2^2l1^2(\cos(q3))^2-2\,l1^2\sin(q3)\cos(q3)l2^2\sin(q4)\cos(q4)} \\ \frac{l2\left(-\sin(q4)-\sin(q4)l1^2(\cos(q3))^2+\cos(q4)l1^2\sin(q3)\cos(q3)\right)}{w^2+l1^2w^2+l1^2l2^2(\cos(q4))^2-2\,l1^2l2^2(\cos(q4))^2(\cos(q3))^2+l2^2+l2^2l1^2(\cos(q3))^2-2\,l1^2\sin(q3)\cos(q3)l2^2\sin(q4)\cos(q4)} \end{bmatrix}$$

>g2:=<Gq[1,2],Gq[2,2],Gq[3,2],Gq[4,2]>;

$g2 :=$

$$\begin{bmatrix} \frac{l1^2\sin(q3)\cos(q3)w^2+l2^2\sin(q4)\cos(q4)}{w^2+l1^2w^2+l1^2l2^2(\cos(q4))^2-2\,l1^2l2^2(\cos(q4))^2(\cos(q3))^2+l2^2+l2^2l1^2(\cos(q3))^2-2\,l1^2\sin(q3)\cos(q3)l2^2\sin(q4)\cos(q4)} \\ -\frac{-w^2-l1^2w^2+l1^2(\cos(q3))^2w^2-l2^2+l2^2(\cos(q4))^2}{w^2+l1^2w^2+l1^2l2^2(\cos(q4))^2-2\,l1^2l2^2(\cos(q4))^2(\cos(q3))^2+l2^2+l2^2l1^2(\cos(q3))^2-2\,l1^2\sin(q3)\cos(q3)l2^2\sin(q4)\cos(q4)} \\ \frac{l1\left(-\sin(q3)l2^2\sin(q4)\cos(q4)+\cos(q3)w^2+\cos(q3)l2^2-\cos(q3)l2^2(\cos(q4))^2\right)}{w^2+l1^2w^2+l1^2l2^2(\cos(q4))^2-2\,l1^2l2^2(\cos(q4))^2(\cos(q3))^2+l2^2+l2^2l1^2(\cos(q3))^2-2\,l1^2\sin(q3)\cos(q3)l2^2\sin(q4)\cos(q4)} \\ -\frac{l2\left(\sin(q4)l1^2\sin(q3)\cos(q3)-\cos(q4)-\cos(q4)l1^2+\cos(q4)l1^2(\cos(q3))^2\right)}{w^2+l1^2w^2+l1^2l2^2(\cos(q4))^2-2\,l1^2l2^2(\cos(q4))^2(\cos(q3))^2+l2^2+l2^2l1^2(\cos(q3))^2-2\,l1^2\sin(q3)\cos(q3)l2^2\sin(q4)\cos(q4)} \end{bmatrix}$$

>dffg1:=
<<diff(g1[1],q1),diff(g1[2],q1),diff(g1[3],q1),diff(g1[4],q1)>|
<diff(g1[1],q2),diff(g1[2],q2),diff(g1[3],q2),diff(g1[4],q2)>|
<diff(g1[1],q3),diff(g1[2],q3),diff(g1[3],q3),diff(g1[4],q3)>|
<diff(g1[1],q4),diff(g1[2],q4),diff(g1[3],q4),diff(g1[4],q4)>>;
>dffg2:=
<<diff(g2[1],q1),diff(g2[2],q1),diff(g2[3],q1),diff(g2[4],q1)>|
<diff(g2[1],q2),diff(g2[2],q2),diff(g2[3],q2),diff(g2[4],q2)>|
<diff(g2[1],q3),diff(g2[2],q3),diff(g2[3],q3),diff(g2[4],q3)>|
<diff(g2[1],q4),diff(g2[2],q4),diff(g2[3],q4),diff(g2[4],q4)>>;
>g3:=dffg2.g1-dffg1.g2;
>dffg3:=<<diff(g3[1],q1),diff(g3[2],q1),diff(g3[3],q1),diff(g3[4],q1)>|
<diff(g3[1],q2),diff(g3[2],q2),diff(g3[3],q2),diff(g3[4],q2)>|
<diff(g3[1],q3),diff(g3[2],q3),diff(g3[3],q3),diff(g3[4],q3)>|
<diff(g3[1],q4),diff(g3[2],q4),diff(g3[3],q4),diff(g3[4],q4)>>;
>g4:=dffg2.g3-dffg3.g2;

G.2 Lie Bracket operation for Planar Space Robot (3-body)

>g1:=<-A12/A11,1,0>;

$$g1 := \begin{bmatrix} -\frac{A12}{A11} \\ 1 \\ 0 \end{bmatrix}$$

>g2:=<-A13/A11,0,1>;

$$g2 := \begin{bmatrix} -\frac{A13}{A11} \\ 0 \\ 1 \end{bmatrix}$$

>a11:=b11+c1*cos(ph1)+c2*cos(ph1+ph2);

$$a11 := b11 + c1 \cos(ph1) + c2 \cos(ph1 + ph2)$$

>a12:=b22+c1*cos(ph1)+c3*cos(ph2);

$$a12 := b22 + c1 \cos(ph1) + c3 \cos(ph2)$$

>a13:=b33+c2*cos(ph1+ph2)+c3*cos(ph2);

$$a13 := b33 + c2 \cos(ph1 + ph2) + c3 \cos(ph2)$$

>A11:=a11+a12+a13;

$$A11 := b11 + 2\,c1 \cos(ph1) + 2\,c2 \cos(ph1 + ph2) + b22 + 2\,c3 \cos(ph2) + b33$$

>A12:=a12+a13;

$$A12 := b22 + c1 \cos(ph1) + 2\,c3 \cos(ph2) + b33 + c2 \cos(ph1 + ph2)$$

>A13:=a13;

$$A13 := b33 + c2 \cos(ph1 + ph2) + c3 \cos(ph2)$$

>g1;

$$\begin{bmatrix} -\frac{b22+c1\ \cos(ph1)+2\,c3\ \cos(ph2)+b33+c2\ \cos(ph1+ph2)}{b11+2\,c1\ \cos(ph1)+2\,c2\ \cos(ph1+ph2)+b22+2\,c3\ \cos(ph2)+b33} \\ 1 \\ 0 \end{bmatrix}$$

>g2;

$$\begin{bmatrix} -\frac{b33+c2\ \cos(ph1+ph2)+c3\ \cos(ph2)}{b11+2\,c1\ \cos(ph1)+2\,c2\ \cos(ph1+ph2)+b22+2\,c3\ \cos(ph2)+b33} \\ 0 \\ 1 \end{bmatrix}$$

>dffg1:=<<diff(g1[1],t),diff(g1[2],t),diff(g1[3],t)>|
<diff(g1[1],ph1),diff(g1[2],ph1),diff(g1[3],ph1)>|
<diff(g1[1],ph2),diff(g1[2],ph2),diff(g1[3],ph2)>>;

>dffg2:=<<diff(g2[1],t),diff(g2[2],t),diff(g2[3],t)>|
<diff(g2[1],ph1),diff(g2[2],ph1),diff(g2[3],ph1)>|
<diff(g2[1],ph2),diff(g2[2],ph2),diff(g2[3],ph2)>>;

>g3:=dffg2.g1-dffg1.g2;

g3 :=

$$\begin{bmatrix} \frac{c2\ \sin(ph1+ph2)}{b11+2\,c1\ \cos(ph1)+2\,c2\ \cos(ph1+ph2)+b22+2\,c3\ \cos(ph2)+b33} \\ +\frac{(b33+c2\ \cos(ph1+ph2)+c3\ \cos(ph2))(-2\,c1\ \sin(ph1)-2\,c2\ \sin(ph1+ph2))}{(b11+2\,c1\ \cos(ph1)+2\,c2\ \cos(ph1+ph2)+b22+2\,c3\ \cos(ph2)+b33)^2} \\ +\frac{-2\,c3\ \sin(ph2)-c2\ \sin(ph1+ph2)}{b11+2\,c1\ \cos(ph1)+2\,c2\ \cos(ph1+ph2)+b22+2\,c3\ \cos(ph2)+b33} \\ -\frac{(b22+c1\ \cos(ph1)+2\,c3\ \cos(ph2)+b33+c2\ \cos(ph1+ph2))(-2\,c2\ \sin(ph1+ph2)-2\,c3\ \sin(ph2))}{(b11+2\,c1\ \cos(ph1)+2\,c2\ \cos(ph1+ph2)+b22+2\,c3\ \cos(ph2)+b33)^2} \\ 0 \\ 0 \end{bmatrix}$$

>Gq:=<<g1>|<g2>|<g3>>;

>Rank(Gq);

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