UNIVERSITY of SIEGEN

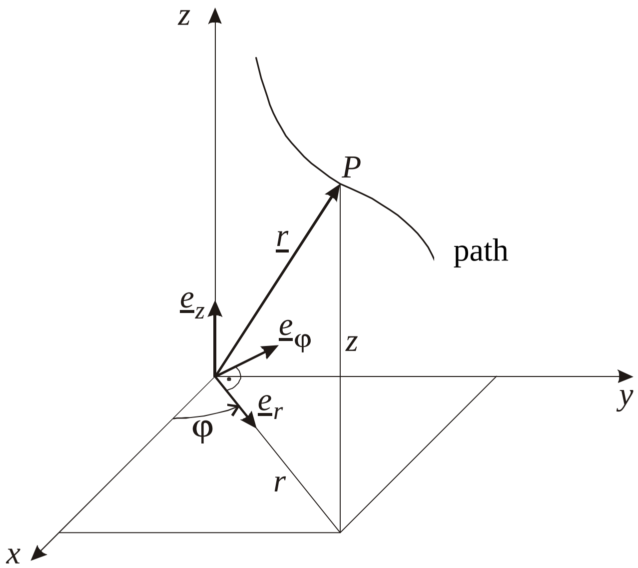
Institute of Mechanics and Automatic Control-Mechatronics
Univ.-Prof. Dr.-Ing. C.-P. Fritzen

Homework 1:

Sometimes it is easier to describe the motion on a curved path in cylindrical coordinates than in cartesian coordinates.
The position vector in cylindrical coordinates is:

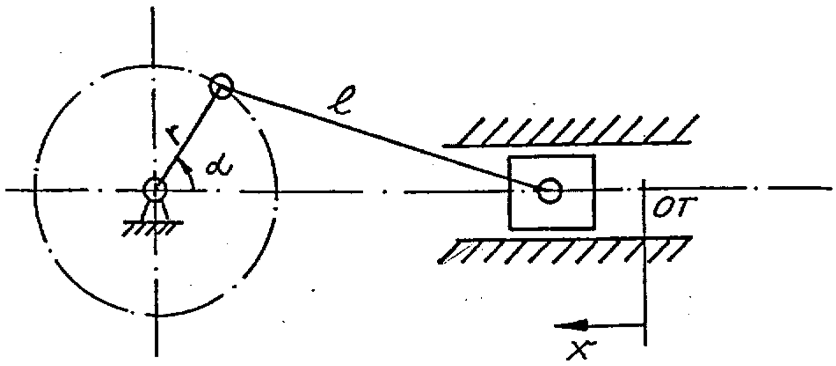
$$\underline{r}(t) = r(t)\,\underline{e}_r + z(t)\,\underline{e}_z$$

whereas the unit vector $\underline{e}_r$ depends on the angle φ and this on his part again depends on the time, which can be written as $\underline{e}_r = \underline{e}_r(\varphi(t))$. Due to this time dependence of the unit vector the derivations with respect to time for the velocity and the acceleration are not as simple expressions as in cartesian coordinates. Determine these expressions.



Institute of Mechanics and Automatic Control-Mechatronics
Univ.-Prof. Dr.-Ing. C.-P. Fritzen

Homework 2:



Given the crank mechanism with a crank shaft with an offset *r*, a connection rod of length *l* and the piston. The position of the piston is given by *x* starting at the dead centre (top dead centre TDC).

1) Determine expressions for the piston position $x(\alpha)$, velocity $v(\alpha)$ and acceleration $a(\alpha)$.

   Assume that the angular velocity $\dot{\alpha} = \omega = const.$, introduce the ratio $\lambda = r/l$.

2) Simplify the expressions for small values of $\lambda = r/l$.

   Hint: use the expansion and truncate the series:
   $$\sqrt{1-\delta} \approx 1 - \frac{\delta}{2} \qquad \text{für} \quad |\delta| \leq 1$$

3) Determine *x*, *v*, *a* for $l \to \infty \quad (\lambda \to 0)$.

UNIVERSITY of SIEGEN

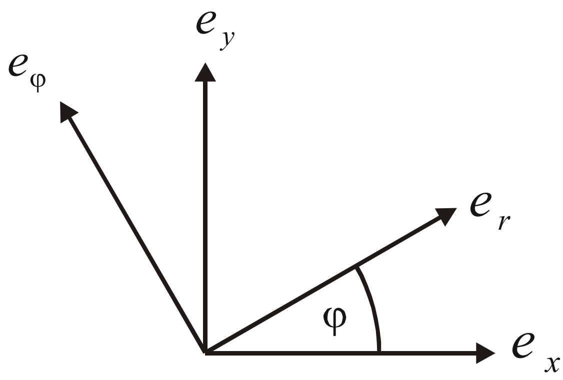
Institute of Mechanics and Automatic Control-Mechatronics
Univ.-Prof. Dr.-Ing. C.-P. Fritzen

Solution of Homework 1:

$$\underline{r}(t) = r(t)\,\underline{e}_r + z(t)\,\underline{e}_z \quad , with \quad \begin{aligned} r(t) &= r \\ z(t) &= z \end{aligned}$$

$$\underline{r} = r\,\underline{e}_r + z\,\underline{e}_z$$

$$\begin{aligned}
\underline{e}_\varphi &= -\sin\varphi\,\underline{e}_x + \cos\varphi\,\underline{e}_y \\
\underline{e}_r &= \cos\varphi\,\underline{e}_x + \sin\varphi\,\underline{e}_y \\
\underline{\dot{e}}_r &= -\dot{\varphi}\sin\varphi\,\underline{e}_x + \dot{\varphi}\cos\varphi\,\underline{e}_y \\
&= \dot{\varphi}\left(-\sin\varphi\,\underline{e}_x + \cos\varphi\,\underline{e}_y\right) \\
&= \dot{\varphi}\,\underline{e}_\varphi \\
\underline{\dot{e}}_\varphi &= -\dot{\varphi}\cos\varphi\,\underline{e}_x - \dot{\varphi}\sin\varphi\,\underline{e}_y \\
&= -\dot{\varphi}\,\underline{e}_r
\end{aligned}$$



Velocity:

$$\begin{aligned}
\underline{v} = \underline{\dot{r}} \quad &= \dot{r}\,\underline{e}_r + r\underline{\dot{e}}_r + \dot{z}\,\underline{e}_z + z\,\underline{\dot{e}}_z \\
&= \dot{r}\,\underline{e}_r + r\,\dot{\varphi}\,\underline{e}_\varphi + \dot{z}\,\underline{e}_z
\end{aligned} \qquad with \quad \underline{\dot{e}}_z = 0$$

$$\begin{aligned}
v_r &= \dot{r} \\
v_\varphi &= r\,\dot{\varphi} \\
v_z &= \dot{z}
\end{aligned}$$

Acceleration:

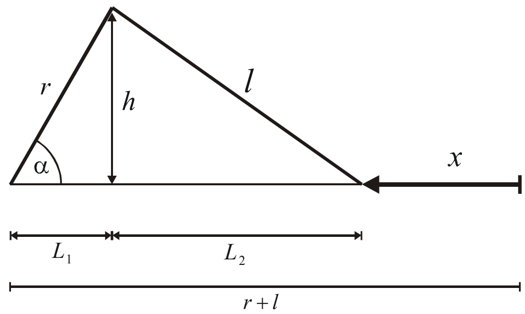
$$\begin{aligned}
a = \underline{\dot{v}} = \underline{\ddot{r}} \quad &= \ddot{r}\,\underline{e}_r + \dot{r}\,\underline{\dot{e}}_r + \dot{r}\,\dot{\varphi}\,\underline{e}_\varphi + r\left(\dot{\varphi}\,\underline{e}_\varphi\right)^{\bullet} + \ddot{z}\,\underline{e}_z + \dot{z}\,\underline{\dot{e}}_z \\
&= \ddot{r}\,\underline{e}_r + \dot{r}\,\dot{\varphi}\;\;\underline{e}_\varphi + \dot{r}\,\dot{\varphi}\,\underline{e}_\varphi + r\left(\ddot{\varphi}\,\underline{e}_\varphi + \dot{\varphi}\,\underline{\dot{e}}_\varphi\right) + \ddot{z}\,\underline{e}_z \\
&= \ddot{r}\,\underline{e}_r + 2\,\dot{r}\,\dot{\varphi}\;\;\underline{e}_\varphi + r\,\ddot{\varphi}\,\underline{e}_\varphi - r\,\dot{\varphi}^2\;\underline{e}_r + \ddot{z}\,\underline{e}_z \\
&= \left(\ddot{r} - r\,\dot{\varphi}^2\right)\underline{e}_r + \left(2\,\dot{r}\,\dot{\varphi} + r\,\ddot{\varphi}\right)\underline{e}_\varphi + \ddot{z}\,\underline{e}_z
\end{aligned}$$

$$\begin{aligned}
a_r &= \ddot{r} - r\,\dot{\varphi}^2 \\
a_\varphi &= 2\,\dot{r}\,\dot{\varphi} + r\,\ddot{\varphi} \\
a_z &= \ddot{z}
\end{aligned}$$

UNIVERSITY of SIEGEN

Institute of Mechanics and Automatic Control-Mechatronics
Univ.-Prof. Dr.-Ing. C.-P. Fritzen

Solution of Homework 2:



1) piston position $x(\alpha)$

$$x=(r+l)-(L_1+L_2)$$

$$L_1 = r\cos\alpha \qquad L_2=\sqrt{l^2-h^2} \qquad h = r\sin\alpha$$

$$\text{with ratio} \quad \lambda = r/l \qquad L_2 = l\sqrt{1-\lambda^2\sin^2\alpha}$$

$$x=x(\alpha)=r\left(1-\cos\alpha\right)+l\left(1-\sqrt{1-\lambda^2\sin^2\alpha}\right)$$

piston velocity $v(\alpha)$

$$v=\dot{x}=\frac{dx}{d\alpha}\frac{d\alpha}{dt}=\frac{dx}{d\alpha}\dot{\alpha}$$

angular velocity $\dot{\alpha}=\omega=const.$

$$\frac{dx}{d\alpha}=r\sin\alpha+\frac{1}{2}l\left(1-\lambda^2\sin^2\alpha\right)^{-\frac{1}{2}}\cdot 2\,\lambda^2\sin\alpha\cos\alpha$$

$$v=\frac{dx}{d\alpha}\dot{\alpha}=r\omega\sin\alpha\left(1+\frac{\lambda\cos\alpha}{\sqrt{1-\lambda^2\sin^2\alpha}}\right)$$

piston acceleration $a(\alpha)$

$$a=\ddot{x}=\dot{v}=\frac{d}{dt}\left(\frac{dx}{d\alpha}\frac{d\alpha}{dt}\right)=\underbrace{\frac{dx}{d\alpha}}_{\underline{\underline{J}}_{rq}}\underbrace{\ddot{\alpha}}_{\underline{\ddot{q}}}+\underbrace{\frac{d^2x}{d\alpha^2}}_{\underline{\underline{K}}_{rq}}\underbrace{\dot{\alpha}^2}_{\underline{\dot{q}}_Q}$$

$$a=\frac{d^2x}{d\alpha^2}\dot{\alpha}^2 \qquad \text{with} \quad \omega=const. \;\Rightarrow\; \ddot{\alpha}=\dot{\omega}=0$$

$$\frac{d^2x}{d\alpha^2} = r\cos\alpha\left(1 + \frac{\lambda\cos\alpha}{\sqrt{1-\lambda^2\sin^2\alpha}}\right) + r\sin\alpha\left(\frac{-\lambda\sin\alpha}{\sqrt{1-\lambda^2\sin^2\alpha}} + \frac{\left(-\frac{1}{2}\right)\lambda\cos\alpha\cdot\left(-2\lambda^2\sin\alpha\cos\alpha\right)}{\left(1-\lambda^2\sin^2\alpha\right)^{3/2}}\right)$$

$$= r\left\{\cos\alpha\left(1 + \frac{\lambda\cos\alpha}{\sqrt{1-\lambda^2\sin^2\alpha}}\right) + \sin\alpha\left(\frac{-\lambda\sin\alpha}{\sqrt{1-\lambda^2\sin^2\alpha}} + \frac{\lambda^3\cos^2\alpha\sin\alpha\cdot}{\left(1-\lambda^2\sin^2\alpha\right)^{3/2}}\right)\right\}$$

$$a = r\omega^2\left\{\cos\alpha\left(1 + \frac{\lambda\cos\alpha}{\sqrt{1-\lambda^2\sin^2\alpha}}\right) + \sin\alpha\left(\frac{-\lambda\sin\alpha}{\sqrt{1-\lambda^2\sin^2\alpha}} + \frac{\lambda^3\cos^2\alpha\sin\alpha\cdot}{\left(1-\lambda^2\sin^2\alpha\right)^{3/2}}\right)\right\}$$

2) small values of λ

$$\begin{aligned} x &= r(1-\cos\alpha) + l\left(1-\sqrt{1-\lambda^2\sin^2\alpha}\right) \\ &= r(1-\cos\alpha) + l\left(1-\sqrt{1-\delta}\right) \\ &\approx r(1-\cos\alpha) + l\left(1-\left(1-\tfrac{\delta}{2}\right)\right) \\ &= r(1-\cos\alpha) + l\tfrac{\delta}{2} \\ &= r\left(1-\cos\alpha + \tfrac{l}{r}\tfrac{\delta}{2}\right) \\ &= r\left(1-\cos\alpha + \tfrac{\delta}{2\lambda}\right) \\ &= r\left(1-\cos\alpha + \tfrac{\lambda}{2}\sin^2\alpha\right) \end{aligned}$$

$$x = r\left(1-\cos\alpha + \tfrac{\lambda}{4} - \tfrac{\lambda}{4}\cos(2\alpha)\right)$$

$$v = r\omega\left(\sin\alpha + \tfrac{\lambda}{2}\sin(2\alpha)\right)$$

$$a = r\omega^2\left(\cos\alpha + \lambda\cos(2\alpha)\right)$$

with $\delta = \lambda^2\sin^2\alpha \le 1$

$$\sqrt{1-\lambda^2\sin^2\alpha} = \sqrt{1-\delta} \approx 1 - \frac{\delta}{2}$$

with $\sin^2\alpha = \frac{1}{2}\left(1-\cos(2\alpha)\right)$

3) $l \to \infty \qquad \lambda = r/l \to 0$

$$x = r(1-\cos\alpha)$$

$$v = r\omega\sin\alpha$$

$$a = r\omega^2\cos\alpha$$