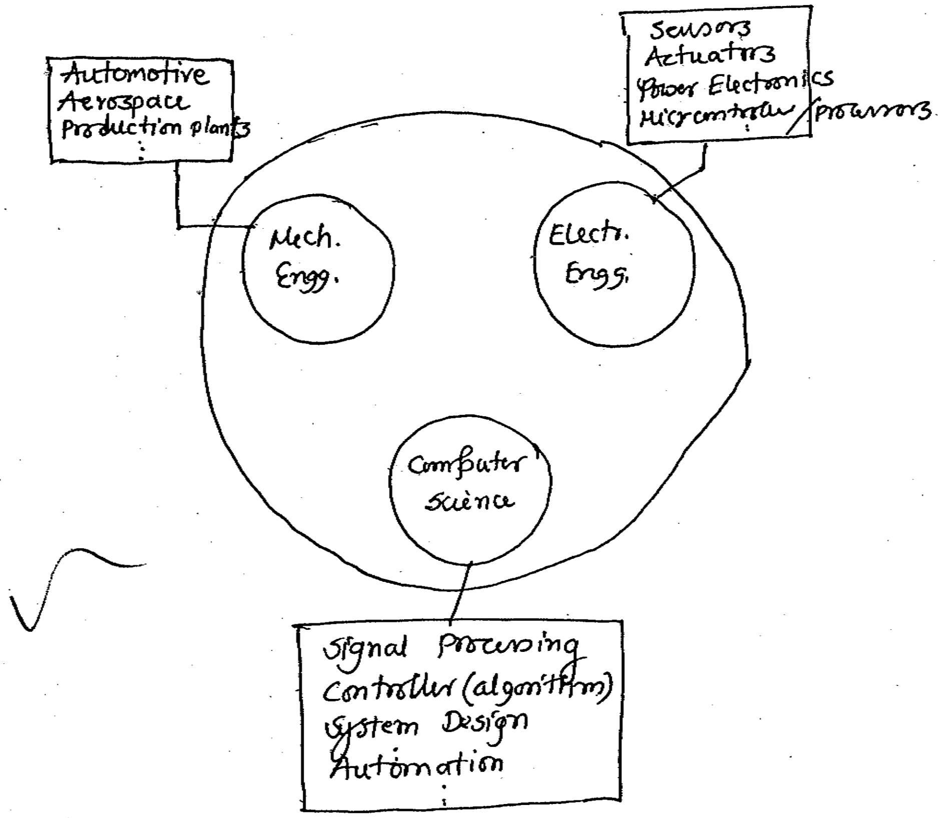
Mechatronic System

21/10/2003

- Introduction
- Modelling
- Identification
- Typical Mechatronic System

1. Introduction:

Mechatronics (1962, Japan)
- Electrical Engineering
- Computer science
- Mechanical Engineering

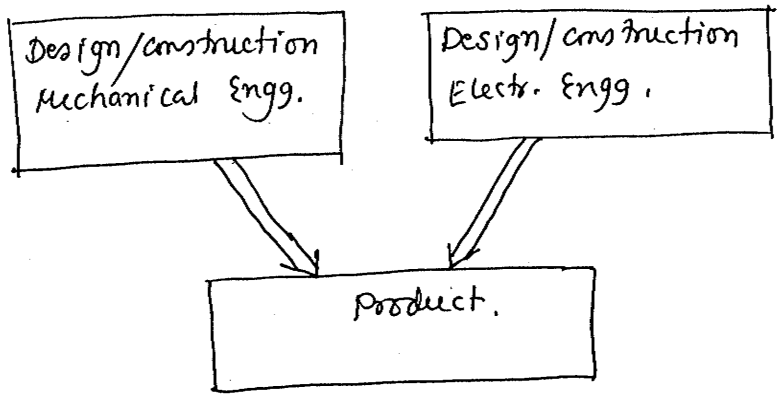


Features: (Additional)
- Human Interface
- telematics
- reduced cost for maintenance

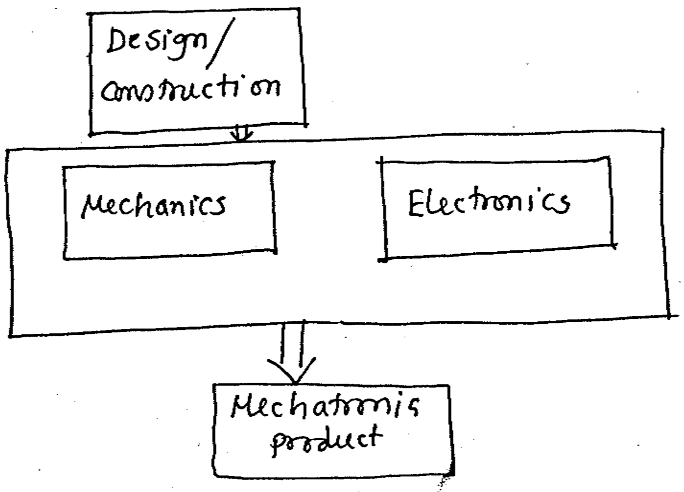
Mechatronics (Definition of IEEE)

Mechatronics is a way of thinking about a product and a system design, that allow engineers to integrate precision mechanical Engineering, Electronics, control Engineering and computer science into the fundamental design process rather than engineering each set of requirements separately.

Traditional Design:



Mechatronical Design:



Additional Aspects
- legal Aspects
- Maintenance
- Human Interfaces
- cost oriented aspects.

but quality for lowest price?

3
- photo-resistor
- metal/hall sensor

} target detection

2. Modelling:

Theoretical — Practical

Theoretical ⇓ Known Principles of Physics, Biology, Chemistry ....etc.

Practical ⇓ used for vallidation
Parameter identification
Structure identification

3.1. Theoretical Modeling:

- Modeling and Simulation are core elements of a mechatronic-design.
- Strategies for modeling

Basic Models:

— balance equations of mass, energy, entropy ....etc.
— Constituent Equations for characterizing physical,... phenomena.
— phenomenological Equations describing irreversible physical, chemical processes ie heat transfer.
— interconnecting Equations.

⇒ Mathematical model: differential Equations (contentrated / partially distributed)
— no. linear
— time variant

mass	steady state	fluid	liquid
			gas
		~~rigid~~ solid	rigid
			elastic
	motion	fluid	liquid
			gas
		~~rigid~~ solid	rigid
			Elastic

Differential Equations:

Energy: mechanical, thermal, electrical, chemical, neclear
— lumped parameter systems
— partial distributed parameter systems

Flows: Main flows and auxiliary flows
usually the dynamics is defined by one main flow!
at least one flow is necessary.

Examples of main flow:

- mechanical : rotational / longitudinal
- electr[redacted]ical : DC/AC
- mass flow
- information

Auxiliary Elements:

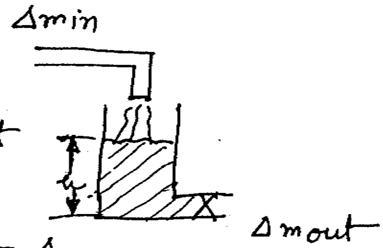
Passive Elements :

- transformator
- Gear
- heat transformer (heat exchanger ?)

Active Elements:

- Electronic Amplifiers
- Voltage generator
- motor

Balance Equation: (Examples)



Mass balance : $\sum m_i =$ constant

time slot : Δt ; $\Delta m_{in} - \Delta m_{out} = \Delta m$

$$\lim_{\Delta t \to 0} \left\{ \frac{\Delta m_{in}}{\Delta t} - \frac{\Delta m_{out}}{\Delta t} = \frac{\Delta m}{\Delta t} \right\}$$

$$\Rightarrow \frac{dm_{in}}{dt} - \frac{dm_{out}}{dt} = \frac{dm}{dt}$$

$$\Rightarrow \boxed{\dot{m}_{in} - \dot{m}_{out} = \dot{m}}$$

http://144.99.38.210/mechatronics

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm}{dt}$$

Energy flow : $\sum E_i = \text{const.}$

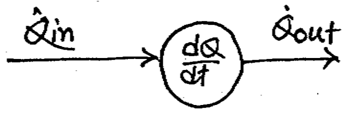
Time: Δt $\quad \Delta E_{in} - \Delta E_{out} = \Delta E$

$$\lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} = \dot{E} = \frac{dE}{dt}$$

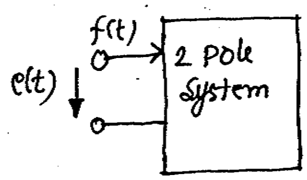
likewise, $\boxed{\dot{E}_{in} - \dot{E}_{out} = \dot{E}}$ Energy balance

Balance Equations in general: $\dot{Q}_{in} - \dot{Q}_{out} = \frac{dQ}{dt}$

change inside the system



Constituent Equations:
They describe the relation between input and output of a system (physical state Equations)



volume flowrate ?

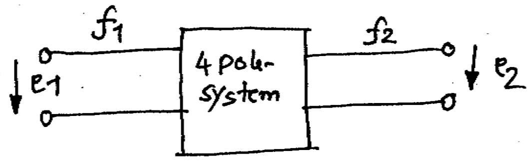
$f(t)$: flow (current, volume, velocity ... ...)

$e(t)$: potential difference (voltage, pressure, force ...)

$p(t) = f(t) \cdot e(t) = \text{Power}$

consumer: e and f in the same direction
source : opposite directions (e and f opposite)

4 pole system:



Energy: $E(t) = \int_0^t P(\tau)\, d\tau$

General Interface variable:

– energy $\quad E(t) = \int_0^t p(\tau)\,d\tau = \int_0^t f(\tau)\, e(\tau)\, d\tau$

– displacement variable $\quad q(t) = \int_0^t f(\tau)\, d\tau$ for example: $f(\tau) = v(\tau) = \dot{x}$

velocity

$$q(t) = \int_0^t \frac{dx}{d\tau} \cdot d\tau = \int_0^t dx$$

displacement

– impulse variable

$$p(t) = \int_0^t e(\tau)\,d\tau$$

সুরঞ্জন সরকার

For example : $e(\tau) = F(\tau)$ (force)

$$p(t) = \int_0^t F(\tau)d\tau = \int_0^t m\underbrace{a(\tau)}_{\frac{dv}{d\tau}}\,d(\tau) = m\int_0^t dv = \underbrace{mv}_{\text{impulse}}$$

For example :- mass flow $\dot{m} = f$

pressure difference $= e$

or temperature difference

– volume flow $= f$

same as mass flow, only for incompressible fluid

$\gamma = \rho g$ ← specific wt

$\dot{m} = \rho\dot{V}$ ← specific weight

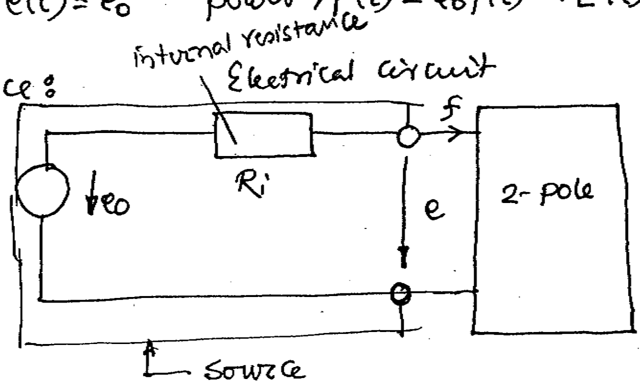
$\rho = \frac{m}{V}$

Sources : produces potential

ideal source : potential doesn't depend on f

$e(t) = e_0$ power, $P(t) = e_0 f(t)$ & $E(t) = e_0\, q(t)$

Real source :



$$e = e_0 - \underbrace{R_i\, f}_{e_f(f)}$$

potential e depends on f

$$e(t) = e_0 - e_f(f)$$

Ideal flow source,

$f(t) = f_0$ $p(t) = f_0\, e(t)$ $E(t) = f_0\, p(t)$

Real flow source :

$f(t) =$ depends on e

$f(t) = f_0 - f_e(e)$

Storage : – potential storage $e(t) = \frac{1}{C}\int_0^t f(\tau)\,d\tau$ (Integral from)

$\dot{e}(t) = \frac{1}{C}\, f(t)$ (differential from) flow

Flow storage: $f(t) = \frac{1}{L}\int_0^t e(\tau)d\tau$ (Integral)

$\dot{f}(t) = \frac{1}{L} e(t)$ (differential)

$$P(t) = e(t)\cdot f(t) = L\dot{f}(t)\, f(t)$$

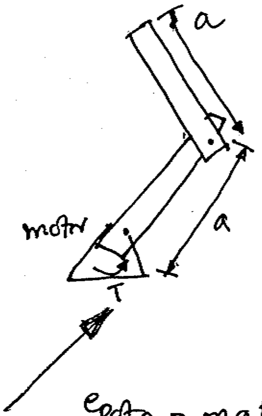
$$E(t) = \int_0^t P(\tau)\,d\tau = \int_0^t L\,\frac{df}{d\tau}\cdot f(\tau)\,d\tau = L\int_{f(0)}^{f(t)} f\cdot df.$$

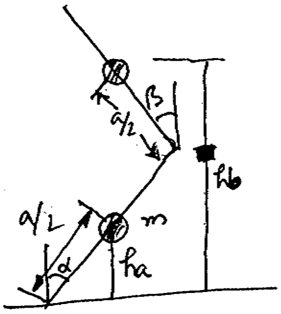
Lec-4

Mechatronics Design

Mr. Kühle

10/11/2003

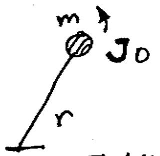




$$e_{pot,a} = m g h_a = m g \frac{a}{2} \cos\alpha$$

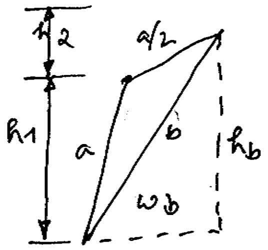
$$e_{pot,b} = m g h_b$$
$$= mg\left(a\cos\alpha + \frac{a}{2}\cos\beta\right).$$

$$\therefore e_{pot} = mg\cdot\left(\frac{3}{2}a\cos\alpha + \frac{a}{2}\cos\beta\right).$$
$$= \frac{1}{2}mg\left(3a\cos\alpha + a\cos\beta\right).$$



$$J = J_0 + m r^2.$$

$$e_{kin,a} = \frac{1}{2} J \dot{\alpha}^2 = \frac{1}{2}\left(J_0 + \frac{1}{4} m a^2\right)\dot{\alpha}^2.$$



$$h_b = h_1 + h_2 = a\cos\alpha + \frac{a}{2}\cos\beta = \frac{a}{2}\left(2\cos\alpha + \cos\beta\right),$$
$$w_b = w_1 + w_2 = \frac{a}{2}\left(2\sin\alpha + \sin\beta\right),$$
$$b^2 = h_b^2 + w_b^2 = \frac{a^2}{4}\left(2\cos\alpha + \cos\beta\right)^2 + \frac{a^2}{4}\left(2\sin\alpha + \sin\beta\right)^2.$$
$$= \frac{a^2}{4}\left(4\cos^2\alpha + 4\cos\alpha\cos\beta + \cos^2\beta + 4\sin^2\alpha + 4\sin\alpha\sin\alpha + \sin^2\beta\right).$$
$$= \frac{a^2}{4}\left[5 + 4\cos(\alpha-\beta)\right]$$
$$= \frac{5}{4}a^2 + a^2\cos(\alpha-\beta).$$

$$e_{kin,b} = \frac{1}{2} J (\dot{\alpha}+\dot{\beta})^2 = \frac{1}{2}\left(J_0 + m\cdot b^2\right)(\dot{\alpha}+\dot{\beta})^2$$
$$= \frac{1}{2}\left(J_0 + \frac{1}{4}a^2 m\left(5 + 4\cos(\alpha-\beta)\right)\right)(\dot{\alpha}+\dot{\beta})^2$$

$$e_{kin} = e_{kin,a} + e_{kin,b}$$

$$= \frac{1}{2}\left(J_0 + \frac{1}{4} m a^2\right)\dot{\alpha}^2 + \frac{1}{2}\left(J_0 + \frac{1}{4} a^2 m \left(5 + 4\cos(\alpha - \beta)\right)\right)(\dot{\alpha} + \dot{\beta})^2,$$

$$= \frac{1}{2}\left(J_0 + \frac{1}{4} m a^2\right)\dot{\alpha}^2 + \frac{1}{2}\left(J_0 + \frac{9}{4} a^2 m\right)(\dot{\alpha} + \dot{\beta})^2.$$

linearized $e_{kin,lin}$

Langrange

$$L = e_{kin} - e_{pot},$$

$$= \frac{1}{2}\left(J_0 + \frac{1}{4} m a^2\right)\dot{\alpha}^2 + \frac{1}{2}\left(J_0 + \frac{9}{4} a^2 m\right)(\dot{\alpha} + \dot{\beta})^2 - \frac{1}{2} m g \left(3\cos\alpha \times a + \cos\beta \times a\right),$$

$$= \frac{1}{2}\left(J_0 + \frac{1}{4} m a^2\right)\dot{\alpha}^2 + \frac{1}{2}\left(J_0 + \frac{9}{4} a^2 m\right)(\dot{\alpha} + \dot{\beta})^2 - 2 m g a$$

$$\frac{\partial L}{\partial \alpha} = 0.$$

$$\frac{\partial L}{\partial \dot{\alpha}} = \frac{1}{\cancel{2}}\left(J_0 + \frac{1}{4} m a^2\right) \times \cancel{2}\dot{\alpha} + \frac{1}{\cancel{2}}\left(J_0 + \frac{9}{4} a^2 m\right) \times \cancel{2}(\dot{\alpha} + \dot{\beta}).$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) = \left(J_0 + \frac{1}{4} m a^2\right)\ddot{\alpha} + \left(J_0 + \frac{9}{4} a^2 m\right)(\ddot{\alpha} + \ddot{\beta})$$

$$Q_\alpha = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha}$$

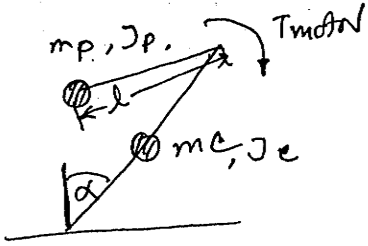
$$= \left(J_0 + \frac{1}{4} m a^2\right)\ddot{\alpha} + \left(J_0 + \frac{9}{4} a^2 m\right)(\ddot{\alpha} + \ddot{\beta}) = T$$

$$\frac{\partial L}{\partial \beta} = 0$$

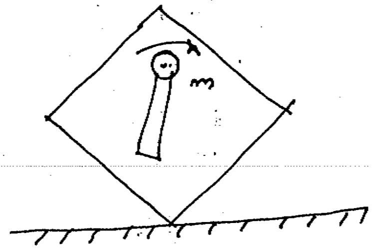
$$\frac{\partial L}{\partial \dot{\beta}} = \frac{1}{\cancel{2}}\left(J_0 + \frac{9}{4} a^2 m\right) \times \cancel{2}(\dot{\alpha} + \dot{\beta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) = \left(J_0 + \frac{9}{4} a^2 m\right)(\ddot{\alpha} + \ddot{\beta})$$

$$Q_\beta = \left(J_0 + \frac{9}{4} a^2 m\right)(\ddot{\alpha} + \ddot{\beta}) = 0$$



- Inverted Pendulum.



- cube

$m_c \rightarrow$ mass of cube
$m_p \rightarrow$ mass of pendulum.

calculate $e_{pot} + e_{kin}$
lagrange equation.

Lec-5

17/11/2003

2.2. Model Simplification

- Experimental modeling
  - Apply test signals
  - measuring the response
  - find a functional relation between input and output.
    (transfer function, fourier analysis, ..)
    step response identification, differential equation and so on...

- Theoretical modeling
  - Find Relations between input-output, input-system states, state-output
    ⇒ complex system Description

  System :
  - Partial Differential Equation
  - non-linear
  - time variant

- Validation of the model by comparing a system response to a test signal between real system and model.
  - improve and adapt the model
  - simplification of the complex model

simplifications:

- Partial system Equations
- non-linear
- time variant

$\xrightarrow{\text{discretization}}$

- system of ordinary differential Eq^n
- non-linear
- time variant

⇓ linearization

- System of ordinary differential eq^n
- linear
- time variant

⇓ time-slot

- System of ordinary differential eq^n
- linear
- time invariant

⇓ or directly

Ordinary

2.2.1. Linearization

- Linearization around the working point
- Linearization along a trajectory

Linearization around a working point (frequency domain, time domain)

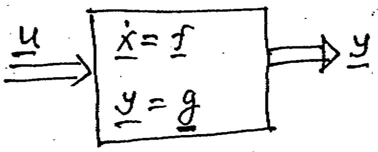
We use it here because f. domain is only applicable for linear system.

$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$ $\underline{x} \rightarrow$ vector of state variables $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\dot{\underline{x}} = \frac{d\underline{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$

$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$ $\underline{u}$ - vector of input variables $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$

Output $\underline{y} = \underline{g}(\underline{x}, \underline{u}, t)$ $\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix}$ $\underline{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_q \end{bmatrix}$



working point: $\dot{\underline{x}} = \underline{0} \Rightarrow (\underline{x}_0 ; \underline{u}_0) \Rightarrow \underline{f}(x_0, y_0, t) = \underline{0}$

Deviation around the working point:

$\Delta \underline{x} = \underline{x} - \underline{x}_0$

$\Delta \underline{u} = \underline{u} - \underline{u}_0$ $\Rightarrow \underline{g}(\underline{x}_0, \underline{y}_0, t) = \underline{y}_0$

$\Delta \underline{y} = \underline{y} - \underline{y}_0$

$\dot{\underline{x}} = (\underline{x}_0 + \Delta \underline{x})^{\cdot} = \underbrace{\cancel{\dot{x}_0}}_{\to 0} + \Delta \dot{\underline{x}} = \underline{f}(\underline{x}_0 + \Delta \underline{x}, \underline{u}_0 + \Delta \underline{u}, t)$

$\underline{y}_0 + \Delta \underline{y} = \underline{g}(\underline{x}_0 + \Delta \underline{x}, \underline{u}_0 + \Delta \underline{u}, t)$

Taylor series: (for scalars) $f(\underbrace{x_0 + \Delta x}_{x}, \underbrace{u_0 + \Delta u}_{u}) = f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0, y_0} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0, y_0} \Delta u +$

$$\underline{f}(\underline{x_0}+\underline{\Delta x}, \underline{u_0}+\underline{\Delta u}) = \begin{bmatrix} f_1(x_{10}+\Delta x_1, x_{20}+\Delta x_2, \dots x_{n0}+\Delta x_n, u_{10}+\Delta u_1, \dots u_{p0}+\Delta u_p) \\ f_2(\qquad) \\ \vdots \\ f_n(\qquad) \end{bmatrix}$$

$$\cong \begin{bmatrix} f_1(\underline{x_0}, \underline{u_0}) + \left.\frac{\partial f_1}{\partial x_1}\right|_{\underline{x_0}, \underline{u_0}} \Delta x_1 + \dots \left.\frac{\partial f_1}{\partial x_n}\right|_{\underline{x_0}, \underline{u_0}} \Delta x_n + \left.\frac{\partial f_1}{\partial u_1}\right|_{\underline{x_0}, \underline{u_0}} \Delta u_1 + \dots + \left.\frac{\partial f_1}{\partial u_p}\right|_{\underline{x_0}, \underline{u_0}} \Delta u_p + \dots \\ \vdots \\ f_n(\underline{x_0}, \underline{u_0}) + \left.\frac{\partial f_n}{\partial x_1}\right|_{\underline{x_0}, \underline{u_0}} \Delta x_1 + \dots \left.\frac{\partial f_n}{\partial x_n}\right|_{\underline{x_0}, \underline{u_0}} \Delta x_n + \left.\frac{\partial f_n}{\partial u_1}\right|_{\underline{x_0}, \underline{u_0}} \Delta u_1 + \dots + \left.\frac{\partial f_n}{\partial u_p}\right|_{x_0, u_0} \Delta u_p \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} f_1(\underline{x_0}, \underline{u_0}) \\ \vdots \\ \vdots \\ f_n(\underline{x_0}, \underline{u_0}) \end{bmatrix}}_{\underline{f}(\underline{x_0}, \underline{u_0}) = \underline{0}} + \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\underline{x_0}, \underline{u_0}} & \left.\frac{\partial f_1}{\partial x_n}\right|_{\underline{x_0}, \underline{u_0}} \\ \vdots & \\ \left.\frac{\partial f_n}{\partial x_1}\right|_{\underline{x_0}, \underline{u_0}} \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_{\underline{x_0}, \underline{u_0}} \end{bmatrix}}_{\substack{n \times n \\ \text{matrix} \to \underline{A}}} \underbrace{\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}}_{\underline{\Delta x}} + \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial u_1}\right|_{\underline{x_0}, \underline{u_0}} & \left.\frac{\partial f_1}{\partial u_p}\right|_{\underline{x_0}, \underline{u_0}} \\ \vdots & \\ \left.\frac{\partial f_n}{\partial u_1}\right|_{\underline{x_0}, \underline{u_0}} \cdots & \left.\frac{\partial f_n}{\partial u_p}\right|_{\underline{x_0}, \underline{u_0}} \end{bmatrix}}_{n \times p \quad \underline{B}} \underbrace{\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \\ \Delta u_p \end{bmatrix}}_{\underline{\Delta u}}$$

$\therefore \; \underline{\dot{\Delta x}} = \underline{A}\,\underline{\Delta x} + \underline{B}\,\underline{\Delta u}$ linearized Differential Equation.

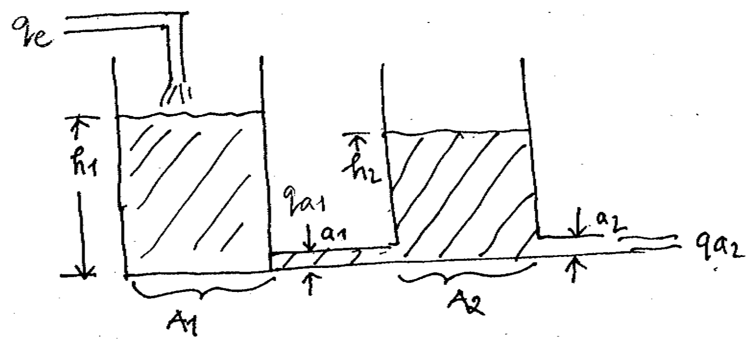
Linearization of output Equation

$$\underline{y}_0 + \Delta\underline{y} = \underline{g}\,\underbrace{(x_0, \underline{u}_0)}_{\underline{y}_0} + \underbrace{\left[\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \cdots\cdots, \frac{\partial g}{\partial x_n}\right]\Bigg|_{x_0, u_0}}_{(q \times n)} \Delta\underline{x}$$

$$+ \left[\frac{\partial g}{\partial u_1}, \frac{\partial g}{\partial u_2}, \cdots \frac{\partial g}{\partial u_n}\right]\Bigg|_{x_0, u_0} \Delta\underline{u}$$

$$\Delta\underline{y} = \underline{\underline{C}} \cdot \Delta\underline{x} + \underline{\underline{D}} \cdot \Delta\underline{u}$$

Example: 2-Tank system



1) volume balance

$V_1 = A_1 h_1$

$\dot{V}_1 = \underbrace{q_e}_{\text{volumeflow}} - q_{a1}$ $\qquad \dot{V}_1 = A_1 \dot{h}_1$

2) $V_2 = A_2 h_2$ $\quad , \dot{V}_2 = A_2 \dot{h}_2$ $\qquad \dot{h}_1 = \frac{\dot{V}_1}{A_1} = \frac{1}{A_1}(q_e - q_{a1})$

$\dot{V}_2 = q_{a1} - q_{a2}$

Torricelli-Law:

$$\left.\begin{aligned} q_{a2} &= a_2\sqrt{2gh_2} \\ q_{a1} &= a_1\sqrt{2g(h_1-h_2)} \end{aligned}\right\} \boxed{\begin{aligned} \dot{h}_1 &= \frac{1}{A_1}q_e - \frac{a_1}{A_1}\sqrt{2g(h_1-h_2)} \\ \dot{h}_2 &= \frac{a_1}{A_2}\sqrt{2g(h_1-h_2)} - \frac{a_2}{A_2}\sqrt{2gh_2} \end{aligned}}$$

2nd order non-linear

state variables: $x_1 = h_1$, $x_2 = h_2$

$u = q_e$; $y = h_2$ (not defined ;)

$$\dot{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{\cdot} = \begin{bmatrix} \frac{1}{A_1} u - \frac{a_1}{A_1}\sqrt{2g(x_1-x_2)} \\ \frac{a_1}{A_2}\sqrt{2g(x_1-x_2)} - \frac{a_2}{A_2}\sqrt{2gx_2} \end{bmatrix} \qquad (1)$$

$y = h_2 = x_2$ $\qquad$ (if $y = q_{a2}$
$y = a_2\sqrt{2gx_2}$)

1) Define a working point : $\underline{\dot{x}} = \underline{0}$

$$0 = \frac{1}{A_1}u_R - \frac{a_1}{A_1}\sqrt{2g(x_{1R}-x_{2R})} \qquad (i)$$

$$0 = \frac{a_1}{A_2}\sqrt{2g(x_{1R}-x_{2R})} - \frac{a_2}{A_2}\sqrt{2gx_{2R}} \qquad (ii)$$

from eqn (2)

$$\frac{a_1^2}{A_2^2} \times \cancel{2g}\,(x_{1R}-x_{2R}) = \frac{a_2^2}{A_2} \times \cancel{2g}\, x_{2R}$$

$$x_{1R} = \frac{a_2^2}{a_1^2} x_{2R} + x_{2R}$$

$$\underline{x_{1R}} = x_{2R}\left(1+\frac{a_2^2}{a_1^2}\right)$$

from eqn (1)

$$\underline{u_R} = a_1\sqrt{2g(x_{1R}-x_{2R})}$$

$$= a_1\sqrt{2g\left(\cancel{x_{2R}} + \frac{a^2}{a_1^2}\times x_{2R} - \cancel{x_{2R}}\right)}$$

$$= \sqrt{\cancel{a_1^2}\times 2g \times \frac{a_2^2}{\cancel{a_1^2}} \times x_{2R}}$$

$$= a_2\sqrt{2g\,x_{2R}} \quad = q_{a2R}$$

2) Linearization :

$$x_1 = x_{1R} + \Delta x_1 \quad ✓$$
$$x_2 = x_{2R} + \Delta x_2 \quad ✓$$
$$u = u_R + \Delta u$$
$$y = y_R + \Delta y$$

$$\underline{\Delta\dot{x}} = \begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_R \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_R \Delta u$$

$$\frac{\partial f_1}{\partial x_1} = -\frac{a_1}{A_1}\cdot\frac{1}{2\sqrt{2g(x_{1R}-x_{2R})}} \times 2g$$

$$\frac{\partial f_1}{\partial x_1} = -\frac{a_1\sqrt{2g}}{2A_1\sqrt{x_{1R}-x_{2R}}} \quad ✓$$

$$\frac{\partial f_1}{\partial x_2} = \frac{a_1}{A_2} \times \frac{+2g}{2\sqrt{2g(x_1-x_2)}} \Rightarrow \left.\frac{\partial f_1}{\partial x_2}\right|_R = \frac{a_1\sqrt{2g}}{2A_1\sqrt{x_{1R}-x_{2R}}} \quad ✓$$

$$\left.\frac{\partial f_2}{\partial x_2}\right|_R = -\frac{a_1\sqrt{2g}}{2A_2\sqrt{x_{1R}-x_{2R}}} - \frac{a_2\sqrt{2g}}{2A_2\sqrt{x_{2R}}} \qquad \left.\frac{\partial f_2}{\partial x_2}\right|_R = \frac{a_1\sqrt{g}}{A_2\sqrt{2(x_{1R}-x_{2R})}} -$$

$$\left.\frac{\partial f_1}{\partial u}\right|_R = \frac{1}{A_1} \qquad \frac{\partial f_2}{\partial u} = 0 \qquad \frac{\partial g}{\partial x_2} = 1$$

Linearization around a trajectory:

System $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u})$

$\underline{y} = \underline{g}(\underline{x}, \underline{u})$

Desired trajectory: $\dot{x}_s = \underline{f}(x_s, u_s)$ $\quad$ $\underline{x_s}, \underline{u_s}, \underline{y_s}$ are not constant!

$\underline{y_s} = g_s(\underline{x_s}, u_s)$ $\quad$ $\dot{x}_s \neq \underline{0}$

To find desired trajectory must not, necessary (to) have a solution!

$$\left.\begin{aligned} \underline{x} &= \underline{x_s} + \Delta \underline{x} \\ \underline{u} &= \underline{u_s} + \Delta \underline{u} \\ \underline{y} &= \underline{y_s} + \Delta y \end{aligned}\right\} \text{ Deviations along trajectory}$$

Taylor Series:

$$(x_s + \Delta \underline{x})^{\cdot} = \dot{\underline{x}}_s + \underline{\Delta \dot{x}} = \underline{f}(\underline{x_s}, \underline{u_s}) + \overbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}^{\substack{(m,n) \\ A(t)}} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

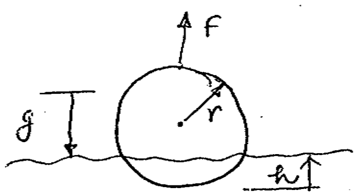
$$+ \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}}_{\underline{B}(t) \quad (n,p)} \Delta u$$

$$\Delta \dot{\underline{x}} = \underline{A}(t) \underline{\Delta} x + \underline{B}(t) \Delta \underline{u}$$

24/11/2003

Mr. Kühle

Example 1:



$F = m\underbrace{u}_{acc.}$

$\dot{v} = g - K(rh^2 - \frac{1}{3}h^3) - u$

$h, u \qquad \dot{v} = f(h, u)$

$h_0 = \frac{1}{3}r \qquad v_0 = 0$

$$\Delta\dot{v} = \frac{\partial f}{\partial h}\Big|_{wp}\cdot\Delta h + \frac{\partial f}{\partial u}\Big|_{wp}\cdot\Delta u + \underbrace{f(h_0, u_0)}_{0}$$

no movement at t_0

$$= \left[0 - K(2hr - h^2)\right]_{h_0, u_0}\cdot\Delta h + \left[0 - 0 - 1\right]_{h_0, u_0}\cdot\Delta u$$

using $h_0 = \frac{1}{3}r$

$$\hookrightarrow \Delta\dot{v} = -K\left(2\times\frac{1}{3}r\cdot r - \frac{1}{9}r^2\right)\Delta h - \Delta u$$

$$= -\underbrace{\frac{5}{9}Kr^2}_{\text{const.} = \text{linear}}\cdot\Delta h - \Delta u.$$

calculate K:

$\dot{v}_0 = 0 = g - K(rh^2 - \frac{1}{3}h^3) - u_0$

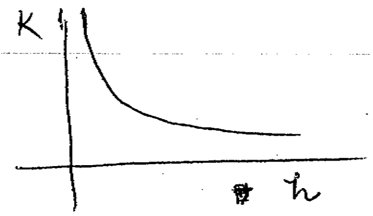
$u_0 = 0$, $K = 10\ g/r^3$

$h_1 = \frac{1}{2}r$

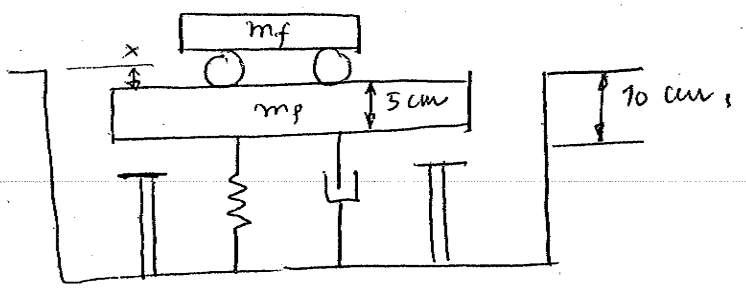
$K = 5\frac{g}{r^3}$

$h_2 = r$

$K = 15\ g/r^3$



Example 2.



$m_p = 1250$ kg $\qquad d = 150000$ kg/s

$c = 10^6$ N/m $\qquad$ if $m_f = 0 \rightarrow x = 0$ m.

(spring const)

$(m_p - m_f)\ddot{x} = m_f g - cx - d\dot{x}$

Input ; m_f
output : x

steady stat : $\dot{x} = 0$ $\ddot{x} = 0$

$(m_p - m_f) \times 0 = m_f g - cx - 0$

$\Rightarrow 0 = m_f g - cx$

$\Rightarrow x = \dfrac{m_f g}{c} = \dfrac{10^{-5}}{} m/kg \cdot m_f$.

$\dfrac{f_{driving}}{f_{driven}} = \dfrac{r_{driving}}{}$

it $x = 2.5$ cm.

$m_f = \dfrac{x}{10^{-5}} = \dfrac{2.5 \times 10^{-2}}{10^{-5}} = 2.5 \times 10^{-2} \times 10^{+5}$
$= 2500$ kg.

$x_1 = x$

$x_2 = \dot{x} = \dot{x}_1 \longrightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = \dfrac{m_f g}{m_p - m_f} - \dfrac{c x_1}{m_p - m_f} - \dfrac{d x_2}{m_p - m_f} = f(x_1, x_2, m_f)$

$\left.\begin{aligned} x_1 &= x_{1,0} + \Delta x_1 \\ x_2 &= x_{2,0} + \Delta x_2 \\ x_3 &= x_{3,0} + \Delta m_f \end{aligned}\right\}$ In steady state
$m_{f,0} = 2500$ kg
$x_{2,0} = \dot{x}_1 = \dot{x} = 0$
$x_{1,0} = x(0) = 0.025$ m.

$\dot{x}_2 = \Delta\dot{x}_2 = \left.\dfrac{\partial f}{\partial x_1}\right|_{x_{1,0}, x_{2,0}, m_{f,0}} \cdot \Delta x_1 + \left.\dfrac{\partial f}{\partial x_2}\right|_{x_{1,0}, x_{2,0}, m_{f,0}} \cdot \Delta x_2 + \left.\dfrac{\partial f}{\partial m_f}\right|_{x_{1,0}, x_{2,0}, m_{f,0}} \cdot \Delta m_f$

$\left.\dfrac{\partial f}{\partial x_1}\right|_{wp} = -\dfrac{c}{m_p - m_f} = 800 \dfrac{1}{s^2}$.

$\left.\dfrac{\partial f}{\partial x_2}\right|_{wp} = -\dfrac{d}{m_p - m_f} = 120 \dfrac{1}{s}$

$x_{2,0} = 0$

$\left.\dfrac{\partial f}{\partial m_f}\right|_{wp} = \dfrac{(m_p - m_{f0})g - m_{f0} g (0-1)}{(m_p - m_{f,0})^2} + \dfrac{c x_{1,0}}{(m_p - m_{f0})^2} + \dfrac{d \cancel{x_{2,0}}}{(m_p - m_{f,0})^2}$

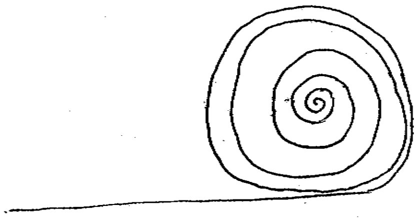
$= \dfrac{1}{(m_p - m_{f,0})^2}\left[m_p g - m_{f0} g - c x_{1,0}\right] = 0.008 \dfrac{m}{kg\, s^2}$.

Prof. Roth Lec-8 25/11/2003

Non-linear

$$\underline{\dot{x}} = f(\underline{x}, u) \longrightarrow \Delta\underline{\dot{x}} = \underline{A}(t)\,\Delta\underline{x} + \underline{B}(t)\,\Delta\underline{u}$$
$$\underline{y} = \underline{g}(\underline{x}, \underline{u}) \longrightarrow \Delta y = \underline{C}(t)\,\Delta\underline{x} + \underline{D}(t)\,\Delta\underline{u}$$
$$(q, n) \qquad (q, p)$$

$$\underline{C}(t) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_q}{\partial x_1} & \cdots & \frac{\partial g_q}{\partial x_n} \end{bmatrix} s \qquad \underline{D}(t) = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_q}{\partial u_p} \\ \vdots & & \\ \frac{\partial g_q}{\partial u_1} & \cdots & \frac{\partial g_q}{\partial u_p} \end{bmatrix} s$$



V = const. moment of Inertia is increasing with time.
Force.

DISCRETIZATION

Simplification of partial Differential Equations by discretization.

$$X(t, \underbrace{x^*, y^*, z^*}_{\text{co-ordinates of a point in 3D.}})$$

$u(t, x^*, y^*, z^*) \longrightarrow$ Partial differential Equations $\longrightarrow X(t, x^*, y^*, z^*)$

General Partial differential eqn:

$$D_t X + D_s X = u(t, x^*, y^*, z^*)$$

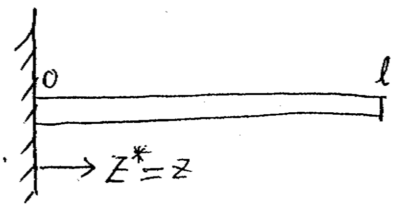
e.g. $\frac{\partial^2 X}{\partial t^2}$

e.g. $\frac{\partial^2 X}{\partial x^*} + \frac{\partial^2 X}{\partial y^*} + \frac{\partial^2 X}{\partial z^*}$

Marginal Conditions: (w.r.t. space) $R_o X = u_{r_o}(t, x_o^*, y_o^*, z_o^*)$
o → original
e → end point
$R_e X = u_{r_e}(t, x_e^*, y_e^*, z_e^*)$

Initial condition (w.r.t. time) $X(0, x^*, y^*, z^*) = x_0(x^*, y^*, z^*)$

Example: 1-dimensional problem of a beam



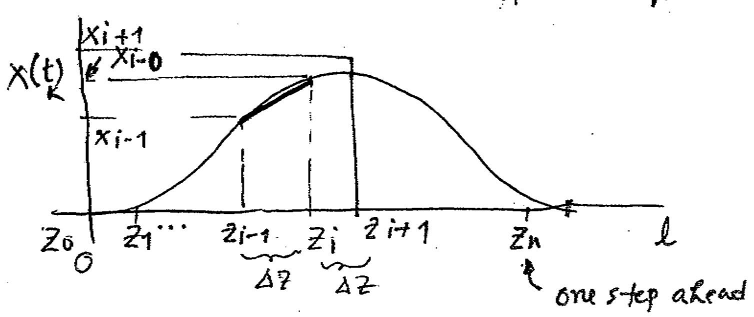
Discretization: Get a set of ordinary differential equations from the partial differential equations:

$$D_s X(t, x^*, y^*, z^*) \longrightarrow D_s X(t, z)$$

$D_s X = \frac{\partial X}{\partial z}$ differential quotient $\xrightarrow{\text{simplify}}$ difference quotient: $\frac{X_i - X_{i-1}}{z_i - z_{i-1}} = \frac{X_i - X_{i-1}}{\Delta z} = \delta_s X$ backward

or $\frac{X_{i+1} - X_i}{\Delta z} = \delta_s X$ forward

$\frac{X_{i+1} - X_{i-1}}{2\Delta z} = \delta_s X$ cent.



$$D_t X(t, z) + D_s X(t, z) = u(t, z)$$

1) at $z = z_i$; $i = 1, \ldots, n$ (inner points)

$$\underbrace{D_t X(t, z_i)}_{\substack{\text{only depends on time} \\ D_t X_i}} + \underbrace{D_s X(t, z_i)}_{\partial_s X_i} = \underbrace{u(t, z_i)}_{u_i(t)}$$

$\Rightarrow D_t X_i = u_i(t) - \partial_s X_i \quad i = 1 \cdots n$ (ordinary differential Eq^n w.r.t. time)

Marginal equations: $R_0 X = u_{\Gamma_0}(t) \xrightarrow{\text{simplify}} \delta_0 X = u_{\Gamma_0}(t)$

$R_e X = u_{\Gamma_e}(t) \xrightarrow{\text{simplify}} \delta_e X = u_{\Gamma_e}(t)$

Initial condition: $X(t_0, z_i) = X(0, z_i) = X_0(z_i)$ $i = 1, 2, \cdots n$

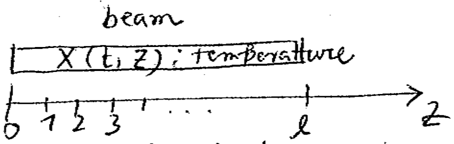
Example: Heat transfer (1D)

Partial differential Eq^n $\frac{\partial x}{\partial t} - \underbrace{\frac{\lambda}{\rho c}}_{1} \cdot \frac{\partial^2 x}{\partial z^2} = 0$

heat capacity (for easier calculation)

Cauchy Marginal condition: $X + \frac{\partial X}{\partial t} = u_0(t) \qquad z = z_0$

$X - \frac{\partial X}{\partial z} = u_e(t) \qquad z = z_e$



$\Delta z = \text{const.} = 0.2\,m \qquad l = 1m.$

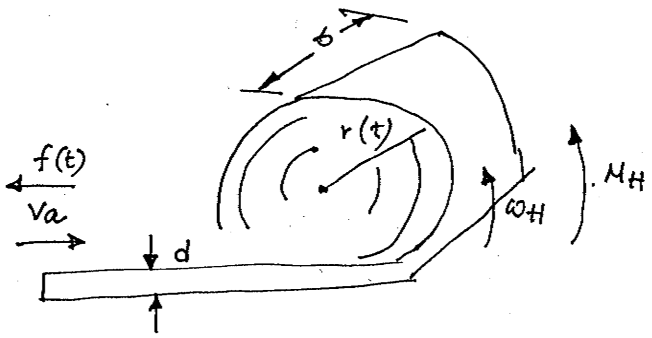
Initial value: $X(0, z) = X_0(z)$

Simplification: $\frac{\partial^2 X}{\partial z^2}$

$$\frac{\partial X^1}{\partial z} = \frac{X_i^1 - X_{i-1}^1}{\Delta z}$$

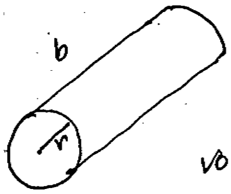
$$\frac{\partial^2 X}{\partial z^2} = \frac{X_i - X_{i-1} - X_{i-1} + X_{i-2}}{(\Delta z)^2}.$$

02/12/2003



$f(t) \rightarrow$ traction force
$Va(t) \rightarrow$ speed of belt
$r(t) \rightarrow$ radius
$\Theta(t) \rightarrow$ inertia
$\omega_H \rightarrow$ angular speed
$M_H \rightarrow$ torque
$d, b, \rho \rightarrow$ thickness, width, density

Task: constant speed Va
Input: M_H

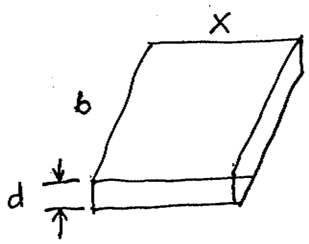


cylinder

volume $V = b \cdot \pi \cdot r^2$

Mass $m(t) = V \cdot \rho = b \rho \pi r^2$

Reel $\frac{dm}{dt} = b\rho\pi \cdot 2r\dot{r}$ (1)



incoming belt

volume $V = bdx$
Mass $m(t) = b\rho dx$

$$\frac{dm}{dt} = b\rho d Va \quad (2)$$

$2b\rho\pi r\dot{r} = b\rho d Va \quad | \; (1) = (2)$ changes of masses is equal

$$\Rightarrow \dot{r} = \frac{b\rho d Va}{2b\rho\pi r}$$

using $v_a = r\omega_H$ of reel $\Rightarrow \dot{r} = \frac{d}{2\pi} \cdot \omega_H$

Energy balance:

Rotational Energy $E_R = \frac{1}{2}\Theta\omega_H^2$

$$\Rightarrow \frac{dE_R}{dt} = \frac{1}{2}\dot{\Theta}\omega_H^2 + \frac{1}{2}\times 2\Theta\omega_H\dot{\omega}_H$$

Moment of Inertia:

$$\Theta(r) = \frac{\pi b\rho}{2} \cdot r^4$$

Incoming Energy of belt:

$E_{kin} = \frac{1}{2}m^* v_a^2$ | m^* incoming mass per time slot with belt

$$\dot{E}_{kin} = \frac{1}{2}\dot{m}^* v_a^2 + m^* v_a \dot{v}_a$$
$$= \frac{1}{2}\underbrace{bd\rho v_a}\cdot v_a^2 + m^* v_a \dot{v}_a$$

$M_H\omega_H - fr\omega_H + \dot{E}_{kin} \overset{!}{=} \dot{E}_R$ Energy change is equal

$$M_H\omega_H - fr\omega_H + \frac{1}{2}bd\rho v_a v_a^2 + m^* v_a \dot{v}_a = \frac{1}{2}\dot{\Theta}\omega_H^2 + \Theta\omega_H\dot{\omega}_H$$

$$\Rightarrow \frac{1}{2}\dot{\Theta}\omega_H + \Theta\dot{\omega}_H = \frac{bd\rho v_a v_a^2}{2\omega_H} + M_H - fr$$

using given $\Theta = \frac{\pi b\rho}{2} r^4$ (3)

$$\hookrightarrow \dot{\Theta} = \frac{\pi}{2} b\rho \times 4r^3 \cdot \dot{r} = 2b\pi\rho r^3\dot{r}$$

$$\Rightarrow \frac{1}{2}\times\frac{\pi}{2}b\rho\times 4r^3\dot{r}\times\omega_H + \frac{\pi b\rho}{2}r^4\times\dot{\omega}_H = \frac{bd\rho v_a v_a^2}{2\omega_H} + M_H - fr$$

using $\dot{r} = \frac{d}{2\pi}\omega_H$ (mass balance)

$$\Rightarrow b\rho r^3 \cdot \frac{d}{2}\omega_H^2 + \frac{1}{2}\pi b\rho r^4\dot{\omega}_H = \frac{1}{2}bd\rho r v_a^2 + M_H - fr$$

$$\Rightarrow \dot{\omega}_H = \frac{d}{\pi r^3}v_a^2 + \frac{2}{\pi b\rho r^4}M_H - \frac{2f}{\pi b\rho r^3} - \frac{d}{\pi r}\omega_H^2$$

using $v_a = r\omega_H$

$$\hookrightarrow \dot{\omega}_H = \frac{2}{\pi b\rho r^4}M_H - \frac{2}{\pi b\rho r^3}f =: f_1$$

$$\dot{r} = \frac{d}{2\pi}\cdot\omega_H =: f_2$$

Non-linear system: linearize around trajectory:

$V_a = V_0 = \text{const}$

$f = f_0$

$V_0 = r_s \cdot \omega_s$

$\hookrightarrow \dot{r}_s = \frac{d}{2\pi} \cdot \omega_s = \frac{d}{2\pi} \cdot \frac{V_0}{r_s}$

$\Rightarrow 2 r_s \dot{r}_s = \frac{V_0 d}{\pi}$

$\Rightarrow \frac{d}{dt}(r_s^2) = \frac{V_0 d}{\pi}$

$\int$

$\Rightarrow r_s^2 = \frac{V_0 d}{\pi} \int_0^t d\tau$

$r_s^2 = \frac{V_0 d}{\pi} \cdot t^0 + C$ at $t=0$ $r_s = r_0$ Radius of axle

$\hookrightarrow C = r_0^2$

$r_s = \sqrt{\frac{V_0 d}{\pi} t + r_0^2}$

$\omega_s = \frac{V_0}{r_s} = \frac{V_0}{\sqrt{\frac{V_0 d}{\pi} t + r_0^2}}$

$\dot{\omega}_s = V_0 \cdot \frac{d}{dt}\left[\left(\frac{V_0 d}{\pi} t + r_0^2\right)^{-\frac{1}{2}}\right]$

$= V_0 \left[\left(\frac{V_0}{\pi} d t + r_0^2\right)^{-3/2} \cdot \frac{V_0 d}{\pi}\right]$

$\Rightarrow \dot{\omega}_s = -\frac{V_0^2 d}{2\pi} \cdot \frac{1}{r_s^3}$

using $\dot{\omega}_s = \frac{2}{b\pi\rho r_s^4} M_s - \frac{2}{\pi b \rho r_s^3} \cdot f_0$

$M_s = \frac{b\pi\rho r_s^4}{2} \dot{\omega}_s + f_0 \cdot r_s$

$M_s = \sqrt{\frac{V_0 d}{\pi} t + r_0^2} \cdot \left(f_0 - \frac{1}{4} b d \rho V_0^2\right)$

should be M_s

$\frac{\partial f_1}{\partial \omega_H} = 0$

$\frac{\partial f_1}{\partial r} = \frac{2}{\pi b \rho}\left(-4 \cdot \frac{M_H}{r^5} + 3 \cdot \frac{f_0}{r_s^4}\right)$ $\qquad \frac{\partial f_2}{\partial r} = 0$

$\frac{\partial f_1}{\partial M_H} = \frac{2}{\pi b \rho r_s^4}$ $\qquad \frac{\partial f_2}{\partial M_H} = 0$

$$\frac{\partial f_1}{\partial f} = -\frac{2}{\pi b \rho r^3}$$

$$\dot{\omega} = \left(\frac{\partial f_1}{\partial \omega}\right)_s \omega_H + \left(\frac{\partial f_1}{\partial r}\right)_s r + \left(\frac{\partial f_1}{\partial M_H}\right)_s M_H + \left(\frac{\partial f_1}{\partial f}\right)_s f$$

$\dot{r}$ = similare

$$\dot{\omega}_H = \frac{2}{\pi b \rho} \cdot \left(3 \frac{f_0}{r_s^4} - 4 \frac{M_s}{r_s^5}\right) r + \frac{2}{\pi b \rho r_s^4} \cdot M_H - \frac{2}{\pi b \rho r_s^3} f$$

$$\dot{r} \Rightarrow \frac{d}{2\pi} \omega_H$$

$r_0 = 0.1 \, m$ $\quad f_0 = 770 \, N$ $\quad b = 1 \, m$ $\quad d = 0.001 \, m$

$V_0 = 10 \, m/s$ $\quad \rho = 7700 \, \frac{kg}{m^3}$

$$M_s = r_0 \left(f_0 - \frac{1}{4} b \rho d V_0^N\right)$$
$$= 57.75 \, Nm$$

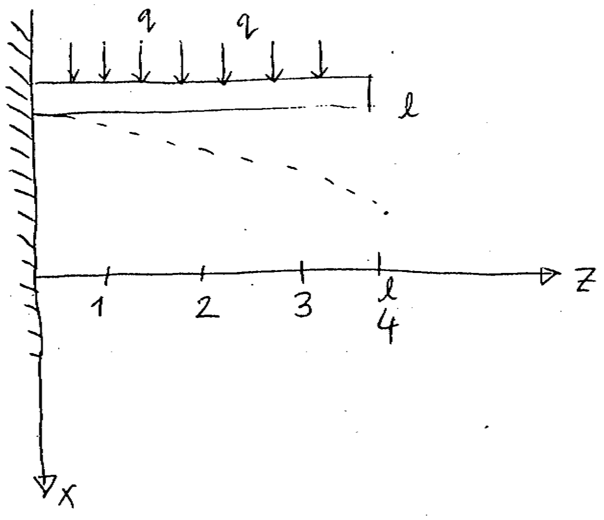
$$\dot{r} = \frac{10^{-3}}{2\pi} \cdot \omega_H$$

$$\dot{\omega_H} = 0.826 \, M_H - 0.0826 \, f$$

Exercise Kühle Lec-10

08/12/2003

Discretization of Bending bar



Divide the bar into 4 equi-distant parts

$D_t X(t_i, z_i) = u(t_i, z_i) - D_z X(t_i z_i)$ in general

$EI X'''' = u(t)$ E = Elasticity
I = Second moment of Inertia
$\hookrightarrow$ EI = stiffness

$$\leftrightarrow 0 = \underbrace{q_i}_{u(t)} - \underbrace{EI X''''}_{D_z X(\ )}$$

$$\Rightarrow 0 = q - EI \cdot \frac{X_{i-2} - 4X_{i-1} + 6X_i - 4X_{i+1} + X_{i+2}}{(\Delta z)^4}$$

outside the bar

$i=0: \boxed{X_{-2} - 4X_{-1}} + 6X_0 - 4X_1 + X_2 = \frac{q \cdot (\Delta z)^4}{EI}$

$i=1: \boxed{X_{-1}} - 4X_0 + 6X_1 - 4X_2 + X_3 = \frac{q(\Delta z)^4}{EI}$

$i=2: X_0 - 4X_1 + 6X_2 - 4X_3 + X_4 = \frac{q(\Delta z)^4}{EI}$

$i=3: X_1 - 4X_2 + 6X_3 - 4X_4 + \boxed{X_5} =$ "

$i=4: X_2 - 4X_3 + 6X_4 \boxed{- 4X_5 + X_6} =$ "

outside the bar

Initial conditions / Marginal equations

$X(0) = X_0 = 0$

$X_0' = 0 \rightarrow \frac{X_{i+1} - X_{i-1}}{2\Delta z} = 0$ $\Rightarrow X_1 - X_{-1} = 0$

$\Rightarrow \frac{X_1 - X_{-1}}{2\Delta z} = 0$

Bending moment:

$$EI \cdot X'' = -M_b$$

Transversal force: $F_q = -EI\,X'''$

Bending moment:

$$EI \cdot \frac{X_{i+1} - 2X_i + X_{i-1}}{(\Delta z)^2} = -M_{b\,i}$$

$$M_b(l) = M_{b4} = 0$$

$$X_{i-1} - 2X_i + X_{i+1} = 0 \quad \Big|\; i=4$$

$$\Rightarrow X_3 - 2X_4 + X_5 = 0$$

Transversal force

$$F_q(l) = F_{q4} = 0 \qquad \frac{-X_{-2} + 2X_{-1} - 2X_{i+1} + X_{i+2}}{2(\Delta z)^3} \begin{array}{l} = F_{q,i} \\ = 0 \text{ at } i=4 \end{array}$$

$$\Rightarrow -X_2 + 2X_3 - 2X_5 + X_6 = 0$$

Using $\Delta Z = \frac{l}{4} \rightarrow (\Delta z)^4 = \frac{l^4}{256}$

1st init. condn

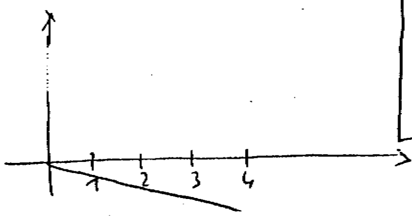
2nd ini. condn

$$\begin{array}{ccccccccc} X_{-2} & X_{-1} & X_0 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \end{array}$$

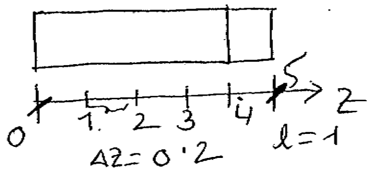
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{256} \\ \frac{1}{256} \\ \frac{1}{256} \\ \frac{1}{256} \\ \frac{1}{256} \\ 0 \\ 0 \end{bmatrix}$$

marginal conditions

$\frac{ql^4}{256}$ * EI



Dr. Roth 08/12/2003



$\Delta z = 0'2$

$$\frac{\partial x}{\partial t} - 1 \cdot \frac{\partial^2 x}{\partial z^2} = 0 \qquad \dot{x}_i = \frac{x_{i+1} - 2i - x_{i-1}}{(\Delta z)^2}$$

For inner points :

$$1)\ \dot{x}_1 - \frac{x_2 - 2x_1 + x_0}{(0'2)^2} = 0$$ [central difference quotient]

$$1)\ \Rightarrow \dot{x}_1 = 25x_2 - 50x_1 + 25x_0$$

$$2)\ \dot{x}_2 = 25x_3 - 50x_2 + 25x_1$$

$$3)\ \dot{x}_3 = 25x_4 - 50x_3 + 25x_2$$

$$4)\ \dot{x}_4 = 25x_5 - 50x_4 + 25x_3$$

Marginal conditions : Cauchy-conditions

$$z = 0 \ ; \ x + \frac{\partial x}{\partial z} = \mu_0(t)$$

$$z = 1 \ ; \ x - \frac{\partial x}{\partial z} = \mu_l(t)$$

$\Downarrow$

Approximation

at $z=0$ forward approximation

$$\frac{\partial x}{\partial z} = \frac{x_1 - x_0}{\Delta z} \rightarrow x_0 + \frac{x_1 - x_0}{\Delta z} = \mu_0(t)$$

$$\Rightarrow x_0 + \frac{x_1 - x_0}{0'2} = \mu_0(t)$$

$$\Rightarrow -4x_0 + 5x_1 = \mu_0(t)$$

$z = l = 1$

$$\frac{\partial x}{\partial z} = \frac{x_5 - x_4}{\Delta z} = x_5 - \frac{x_5 - x_4}{0'2} = u_l(t)$$

$$\Rightarrow -4x_5 + 5x_4 = u_l(t)$$

Substitute x_0 in 1)

$$\dot{x}_1 = 25x_2 - 50x_1 + 25\left(\frac{5}{4}x_1 - \frac{1}{4}u_0\right)$$

$$\Rightarrow \dot{x}_1 = 25x_2 - \frac{75}{4}x_1 - \frac{25}{4}u_0$$

4) $\dot{x}_4 = 25\left(\frac{5}{4}x_4 - \frac{1}{4}u_1\right) - 50x_4 + 25x_3$

$$\dot{x}_4 = -\frac{75}{4}x_4 + 25x_3 - \frac{25}{4}u_l$$

other equations remains unchanged.

State space form:

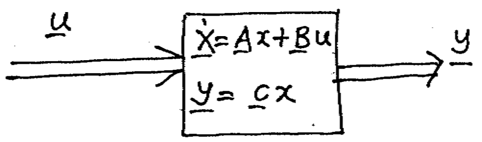
$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\frac{75}{4} & 25 & 0 & 0 \\ 25 & -50 & 25 & 0 \\ 0 & 25 & -50 & 25 \\ 0 & 0 & 25 & -\frac{75}{4} \end{bmatrix} \underline{x} + \underbrace{\begin{bmatrix} -\frac{25}{4} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{25}{4} \end{bmatrix}}_{4\times 2} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$$

Lec-12

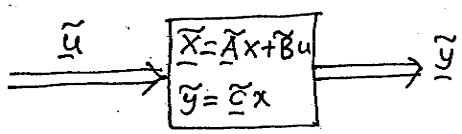
09/12/2003

Order Reduction Techniques:



original system (High order)

⇓ order Reduction : state $\tilde{x}$ has less components than state $\underline{x}$



reduced order system (approximation)

$\underline{A}, \underline{B}, \underline{C}$ are known ; $\tilde{A}, \tilde{B}, \tilde{C}$ are unknown!

Methods for order reduction:
- frequency domain methods (restricted to low order with a few inputs and outputs)
- time domain methods

output error minimization $J = \int_0^\infty (\underline{y}-\tilde{\underline{y}})^T(\underline{y}-\tilde{\underline{y}})\,dt$

this requires numerical optimization

· modal approaches:

Original Model: $\underset{(n)}{\dot{\underline{x}}} = \underset{(n,n)}{\underline{A}}\,\underset{n}{\underline{x}} + \underset{(n,p)}{\underline{B}}\,\underset{p}{\underline{u}}$

$$\underset{q}{\underline{y}} = \underset{(q\times n)}{\underline{C}}\,\underset{(n)}{\underline{x}}$$

⇒ order Reduction : $\dot{\underline{\tilde{x}}} = \underline{\tilde{A}}\,\underline{\tilde{x}} + \underline{\tilde{B}}\,\underline{u}$

$n \to m$ $\quad m \quad (m,m)(m)\ (m,p)(p)$

$(m < n)$ $\quad \underline{\tilde{y}} = \underline{\tilde{C}}\,\underline{\tilde{x}}$

$(m \geq q)$ $\quad q \quad (q,m)\ (m)$

$(m \geq p)$

Again, Objective : $\underline{\tilde{y}}$ shall be a good approximation of $\underline{y}$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \hline \underline{x}_2 \end{bmatrix} \qquad \underline{y} = \underbrace{\left[\begin{array}{c|c} \underline{C}_1 & \underline{C}_2 \end{array}\right]}_{(q,m)} \begin{bmatrix} \underline{x}_1 \\ \hline \underline{x}_2 \end{bmatrix}$$

Which state variables have to be in $\underline{x}_1$?

All state variables which influence $\underline{y}$ $\Rightarrow \underline{C}_2 = 0$

$$\begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \end{bmatrix} = \left[\begin{array}{c|c} \overset{(m,m)}{\underline{A}_{11}} & \overset{(m,n-m)}{\underline{A}_{12}} \\ \hline \underset{(n-m,m)}{\underline{A}_{21}} & \underset{(n-m,n-m)}{\underline{A}_{22}} \end{array}\right] \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \begin{bmatrix} \overset{(m,p)}{\underline{B}_1} \\ \hline \underset{(n-m,p)}{\underline{B}_2} \end{bmatrix} \underline{u}$$

$$\underline{y} = \left[\begin{array}{c|c} \underline{C}_1 & 0 \end{array}\right] \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$$

$$\dot{\underline{x}}_1 = \underline{A}_{11}\underline{x}_1 + \underline{A}_{12}\underline{x}_2 + \underline{B}_1\underline{u}$$

$$\dot{\underline{x}}_2 = \underline{A}_{21}\underline{x}_1 + \underline{A}_{22}\underline{x}_2 + \underline{B}_2\underline{u}$$

1st approximation :
Set $\underline{x}_2 = 0 \longrightarrow$ reduced order, but worse approximation, because complete influence of $\underline{x}_2$ is missing. So, we can not take it.

2nd approximation : Assume the transient behavior of $\underline{x}_2$ is much faster than $\underline{x}_1 \longrightarrow$ Approximate $\underline{x}_2$ by steady state $(\dot{\underline{x}}_2 = \underline{0})$

$$\tilde{\underline{x}}_2 = -\underline{A}_{22}^{-1}\underline{A}_{21}\tilde{\underline{x}}_1 - \underline{A}_{22}^{-1}\underline{B}_2\underline{u}$$

Substitute $\underline{x}_2$: $\dot{\tilde{\underline{x}}}_1 = \underbrace{(\underline{A}_{11} - \underline{A}_{12}\underline{A}_{22}^{-1}\underline{A}_{21})}_{\tilde{A}}\tilde{\underline{x}}_1 + \underbrace{(\underline{B}_1 - \underline{A}_{12}\underline{A}_{22}^{-1}\underline{B}_2)}_{\tilde{B}}\underline{u}$

Reduced order Model $\quad \dot{\tilde{\underline{x}}} = \tilde{\underline{A}}\tilde{\underline{x}} + \tilde{\underline{B}}\underline{u}$

$$\tilde{\underline{y}} = \underline{C}_1\tilde{\underline{x}}_1 \to \tilde{\underline{y}} = \tilde{\underline{C}}\tilde{\underline{x}}.$$

Problem : Assumption is not usually given!
↳ because $\underline{x}_1$ has been chosen to be present in y and not that the fast variables are in $\underline{x}_2$!

Dynamics of a state variable contents slow and fast parts variables. Variables which are only fast, are not available.

3rd approximation

original system in Jordan Canonical form:

$$\underset{(n)}{\dot{\underline{z}}} = \underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots \lambda_m & & \\ & & \lambda_{m+1} & \\ 0 & & & \ddots \lambda_n \end{bmatrix}}_{\underline{\lambda}} \underline{z} + \underline{B}^* \underline{u}$$

$\lambda_1, \dots \lambda_n$ are the Eigenvalues, all other elements are zero.

(Single Eigen values)

$$\dot{\underline{z}} = \underset{(n,n)}{\underline{\lambda}}\ \underset{n}{\underline{z}} + \underset{(n,p)}{\underline{B}^*}\ \underset{(p)}{\underline{u}}$$

for one state variable: $\dot{z}_i = \lambda_i z_i + \underline{b}_i^{*T} \underline{u}$ line i of B^*

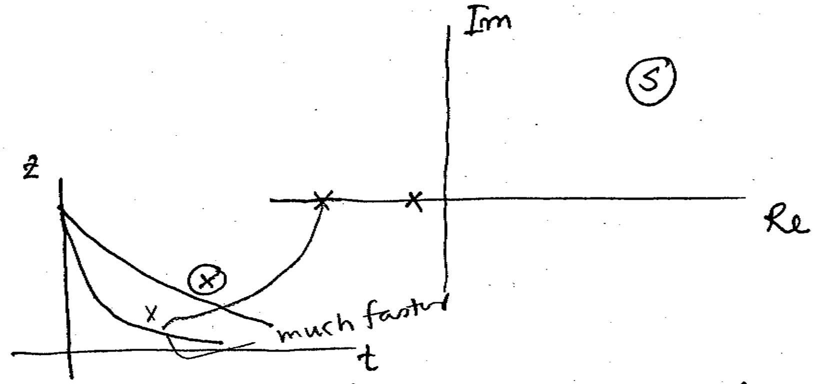
Homogeneous eqn and particular soln

$\Downarrow$

$\dot{z}_i - \lambda_i z_i = 0$ defines the transient behaviour!

$z_i(t) = K_i e^{\lambda_i t}$

If an eigenvalue λ_i is ~~farther~~ far left in the pole plane



if an eigenvalue λ_i is far left in the pole-plane than the related eigen-movement z_i in fast!

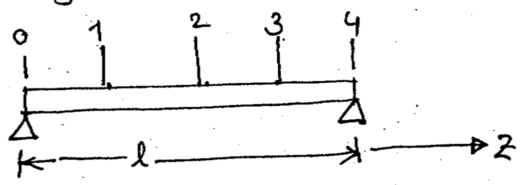
UML Diagram of our classes
class Hemy
class Shark class seer
8/4 or 2 classes
Bib. Curson
Google Search

Lec - 15
Mechatronics
Exercise

16 /12/2003

Discretization

Bending Bar



$$X'' = -\frac{M_b}{K}$$

$$X' = -\frac{M_b}{K}z + C_1$$

$$\Rightarrow X = -\frac{M_b}{K}z^2 + C_1 x + C_2$$

$\boxed{K \cdot x'''' = q(z)}$ Bending Equation

$K \cdot X'' = -M_b$

$-K \cdot X''' = F_T$

— Difference Quotient
— Eqn. for all discretization points $i = 0, \dots 4$
— find marginal/initial cond.

$$0 = q_i - K X''''$$

$$\Rightarrow 0 = q - K \cdot \frac{X_{i-2} - 4X_{i-1} + 6X_i - 4X_{i+1} + X_{i+2}}{(\Delta z)^4}$$

$i = 0$: $\underbrace{\boxed{X_{-2} - 4X_{-1}}}_{\text{outside}} + 6X_0 - 4X_1 + X_2 = \frac{q \cdot (\Delta z)^4}{K}$

$i = 1$: $\boxed{X_{-1}} - 4X_0 + 6X_1 - 4X_2 + X_3 = \frac{q(\Delta z)^4}{K}$

$i = 2$: $X_0 - 4X_1 + 6X_2 - 4X_3 + X_4 =$ "

$i = 3$: $X_1 - 4X_2 + 6X_3 - 4X_4 + \boxed{X_5} =$ "

$i = 4$: $X_2 - 4X_3 + 6X_4 - \underbrace{\boxed{4X_5 + X_6}}_{\text{outside}} =$ "

Marginal ~~Initial~~ Conditions :

$X(0) = X_0 = 0$

$X_0' = 0 \rightarrow \frac{X_{i+1} - X_{i-1}}{2\Delta z} = 0$

$\Rightarrow X_1 = X_{-1}$

$x_{-1} - x_1 = 0$

$X(4) = X_4 = 0$

$X_4' = 0 \rightarrow \frac{X_{i+1} - X_{i-1}}{2\Delta z} = 0$

$\Rightarrow X_3 - X_5 = 0$

$\Rightarrow X_3 = X_5$

$x_3 - x_5 = 0$

Bending moment

KX

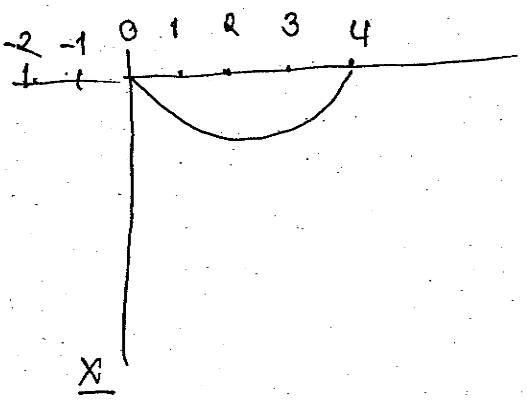
$$
\begin{array}{r|ccccccccc}
 & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & x_3 & x_4 & \cancel{x_5} & \cancel{x_6} \\
\hline
x_0=0 \quad -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{-1}-x_1=0 \quad -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -4 & 6 & 4 & 1 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & -4 & 6 & 4 & 1 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & -4 & 6 & 4 & 1 & 0 \\
4 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & 4 & 1 \\
x_3-x_5=0 \quad 5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
x_4=0 \quad 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
\quad \underline{x} = \begin{bmatrix} 0 \\ 0 \\ \frac{q(\Delta z)^4}{K} \\ \vdots \\ \frac{q(\Delta z)^4}{K} \\ 0 \\ 0 \end{bmatrix}
$$

$\times\times \underbrace{0 \;\; 1 \;\; 2 \;\; 3 \;\; 4} \times\times$

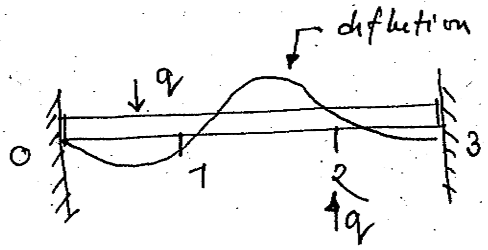
$$\frac{q l^4}{K} = 1$$

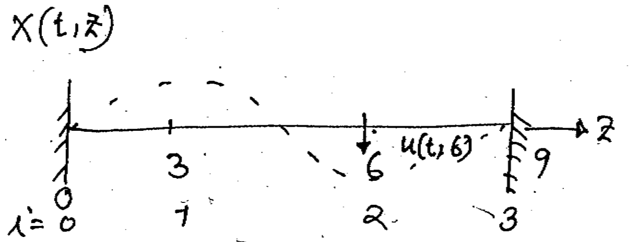
$$
\underline{X} = \begin{bmatrix} 0.0195 \\ 0.0024 \\ \hline 0 \\ 0.0024 \\ 0.0039 \\ 0.0024 \\ 0 \\ \hline 0.0024 \\ 0.0195 \end{bmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\} \text{not applicable} \\ \\ \\ \\ \\ \\ \\ \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\} \text{Not applicable} \end{matrix}
$$

outside



another bar:





String/chord

$$\frac{\partial^2 x}{\partial t^2} - \frac{\partial^2 x}{\partial z^2} = u(t,z) \qquad \Delta z = 3$$

$$\leftrightarrow \frac{\partial^2 x}{\partial t^2} = \underbrace{u(t,z)}_{u_i(t)} + \frac{\partial^2 x}{\partial z^2}$$

$$\Rightarrow \frac{\partial^2 x}{\partial z^2} \approx \frac{x_{i-1} - 2x_i + x_{i+1}}{(\Delta z)^2} \rightarrow \ddot{x}_1 = \frac{1}{9}x_0 - \frac{2}{9}x_1 + \frac{1}{9}x_2$$

$$\ddot{x}_2 = \frac{1}{9}x_1 - \frac{2}{9}x_2 + \frac{1}{9}x_3 + u(t)$$

$x(t,0) = 0$ $\quad$ $\dot{x}_0 = \frac{\partial x(t,0)}{\partial t} = 0$

$x(t,3) = 0$ $\quad$ $\dot{x}_3 = 0$ $\Big\}$ not used

value of i

$$\ddot{x}_1 = \frac{1}{9}x_2 - \frac{2}{9}x_1$$

$$\ddot{x}_2 = \frac{1}{9}x_1 - \frac{2}{9}x_2 + u(t)$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

$y_1 = x_1$

$y_2 = x_2$

$$\dot{y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2}{9} & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{9} & 0 & -\frac{2}{9} & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t,6)$$

$$\ddot{x}_1 = \dot{y}_2 \Rightarrow \dot{y}_2 = \frac{1}{9}y_3 - \frac{2}{9}y_1$$

$$\ddot{x}_2 = \dot{y}_4 \Rightarrow \dot{y}_4 = \frac{1}{9}y_1 - \frac{2}{9}y_3 + u(t)$$

$$\dot{y}_1 = \dot{x}_1 = y_2$$

$$\dot{y}_3 = \dot{x}_2 = y_4$$

Prof. Roth Lec-14 06/01/2004

High order system (n)

$$\underset{(n)}{\dot{\underline{x}}} = \underset{(n,n)}{\underline{A}}\,\underline{x} + \underset{(n,p)}{\underline{B}}\,\underset{(p)}{\underline{u}}$$

linear time-invariant

$$\underset{(q)}{\underline{y}} = \underset{(q,n)}{\underline{C}}\,\underset{(n)}{\underline{x}}$$

Reduced order system (m)

$$\underset{(m)}{\dot{\tilde{\underline{x}}}} = \underset{(m,m)}{\tilde{\underline{A}}}\,\underset{(m)}{\tilde{\underline{x}}} + \underset{(m,p)}{\tilde{\underline{B}}}\,\underset{(p)}{\underline{u}}$$

linear time-invariant

$m \ll n$

$$\underset{(q)}{\tilde{\underline{y}}} = \underset{(q,m)}{\tilde{\underline{C}}}\,\underset{(m)}{\tilde{\underline{x}}}$$

To find $\tilde{\underline{A}}, \tilde{\underline{B}}, \tilde{\underline{C}}$ in order to minimize $\underline{y} - \tilde{\underline{y}}$

Possibilities: $\int_{t_0}^{t_e} \underline{y} - \tilde{\underline{y}}\, dt$

$\int_{t_0}^{t_e} (\underline{y} - \tilde{\underline{y}})^T (\underline{y} - \tilde{\underline{y}})\, dt$ Numerical solutions!

Modal Approach: Eigenvalues of the low-order system is a subset of the Eigenvalues of the higher order system.

$\Rightarrow$ Reduced order system shall have the dominant Eigenvalues of the high order system.

dominant Eigenvalues: eigenvalues, which are near to the imaginary axis, slow dynamics!

$\Rightarrow$ Assumption: If the low-order system has the dominant eigenvalues, $\underline{y} - \tilde{\underline{y}}$ shall be small, means $\tilde{\underline{y}}$ should be a good approximation of $\underline{y}$

state variables

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ \hline x_{m+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \hline \underline{x}_2 \end{bmatrix}$$

<u>How to find m?</u>

m depends on the number of dominant Eigenvalues!

- all state variables, which have influence on $\underline{y}$ (original system) should be represented in the reduced order system.
  - ↳ idea to m
  - ↳ decides which state variables of $\underline{x}$ are in $\underline{x}_1$

$$\Rightarrow \underline{y} = \underline{C}\,\underline{x} = \underline{C}\begin{bmatrix}\underline{x}_1\\ \hline \underline{x}_2\end{bmatrix} = \left[\underline{C}_1 \,\vdots\, \underline{C}_2\right]\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\end{bmatrix} = \left[\underline{C}_1 \,\vdots\, \underline{0}\right]\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\end{bmatrix}$$

- $\tilde{\underline{C}} = \underline{C}_1$
- $\tilde{\underline{x}} = \text{app.}\ \underline{x}_1 = \tilde{\underline{x}}_1$

$$\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\end{bmatrix}^{\bullet} = \underbrace{\left[\begin{array}{c|c}\underset{(m,m)}{A_{11}} & \underset{(m,\,n-m)}{A_{12}}\\ \hline \underset{(n-m,\,m)}{A_{21}} & \underset{(n-m)\,(n-m)}{A_{22}}\end{array}\right]}_{A}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\end{bmatrix} + \underbrace{\left[\begin{array}{c}\underset{(m,p)}{\underline{B}_1}\\ \hline \underset{(n-m,\,p)}{\underline{B}_2}\end{array}\right]}_{\underline{B}}\underline{u}$$

$\underline{x} = \underline{V}\,\underline{z}$

matrix of Eigen vectors

$$\dot{\underline{x}}_1 = \underline{A}_{11}\underline{x}_1 + \underline{A}_{12}\underline{x}_2 + \underline{B}_1\underline{u}$$

$$\dot{\underline{x}}_2 = \underline{A}_{21}\underline{x}_1 + \underline{A}_{22}\underline{x}_2 + \underline{B}_2\underline{u}$$

Jordan-Canonical Form:
(single Eigen values)

$$\underset{(n)}{\dot{\underline{z}}} = \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}\underline{z} + \underline{B}^*\underline{u} \qquad \underline{B}^* = \begin{bmatrix}\underline{B}_1^*\\ \hline \underline{B}_2^*\end{bmatrix}$$

column l is eigenvector l and belongs to eigen value l!

in JCF Eigenvalues are obvious and can be arranged in a group of dominant and non-dominant:

$$\underline{\Lambda} = \left[\begin{array}{c|c}\overbrace{\begin{matrix}\lambda_1 & \\ & \ddots\ \lambda_m\end{matrix}}^{\Lambda_1} & 0\\ \hline 0 & \begin{matrix}\lambda_{m+1} & \\ & \ddots\ \lambda_n\end{matrix}\end{array}\right] = \begin{bmatrix}\underline{\Lambda}_1 & 0\\ 0 & \underline{\Lambda}_2\end{bmatrix}$$

JCF: $\dot{\underline{z}}_1 = \underline{\Lambda}_1\underline{z}_1 + \underline{B}_1^*\underline{u}$

$\dot{\underline{z}}_2 = \underline{\Lambda}_2\underline{z}_2 + \underline{B}_2^*\underline{u} \longrightarrow$ quick part of the dynamics $\dot{\underline{z}}_2 \simeq \underline{0}$

$\downarrow$

$\underline{z}_2 = -\underline{\Lambda}_2^{-1}\underline{B}_2^*\underline{u}$ $(\longrightarrow)$

Prof. Roth

From $\underline{X} = \underline{V}\,\underline{Z} \longrightarrow \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \underline{V}_{11} & \underline{V}_{12} \\ \hline \underline{V}_{21} & \underline{V}_{22} \end{array}\right]}_{\underline{V}} \begin{bmatrix} \underline{Z}_1 \\ \vdots \\ \underline{Z}_2 \end{bmatrix}$

$$\underline{X}_1 = \underline{V}_{11}\underline{Z}_1 + \underline{V}_{12}\underline{Z}_2$$

$$\underline{X}_2 = \underline{V}_{21}\underline{Z}_1 + \underline{V}_{22}\underline{Z}_2$$

$$\underline{Z}_1 = \underline{V}_{11}^{-1}\underline{X}_1 - \underline{V}_{11}^{-1}\underline{V}_{12}\underline{Z}_2$$

$$\underline{X}_2 = \underline{V}_{21}(\underline{V}_{11}^{-1}\underline{X}_1 - \underline{V}_{11}^{-1}\underline{V}_{12}\underline{Z}_2) + \underline{V}_{22}\underline{Z}_2$$

$$\Rightarrow \underline{\tilde{X}}_2 = \underline{V}_{21}\underline{V}_{11}^{-1}\underline{X}_1 - (\underline{V}_{22} - \underline{V}_{21}\underline{V}_{11}^{-1}\underline{V}_{12})\,\underline{\Lambda}_2^{-1}\,\underline{B}_2^{*}\,\underline{u} \qquad (\rightarrow)$$

From $\underline{\dot{X}}_1 = \underline{A}_{11}\underline{X}_1 + \underline{A}_{12}\underline{X}_2 + \underline{B}_1\underline{u}$

$$\Rightarrow \dot{\underline{\tilde{X}}}_1 = \underbrace{(\underline{A}_{11} + \underline{A}_{12}\underline{V}_{21}\underline{V}_{11}^{-1})}_{\tilde{\underline{A}}}\underline{\tilde{X}} + \underbrace{(\underline{B}_1 - \underline{A}_{12}(\underline{V}_{22} - \underline{V}_{21}\underline{V}_{11}^{-1}\underline{V}_{12})\underline{\Lambda}_2^{-1}\underline{B}_2^{*}]}_{\tilde{\underline{B}}}\,\underline{u}$$

$$\dot{\underline{\tilde{X}}}_1 = \underline{A}_{11}\underline{X}_1 + \underline{A}_{12}\underline{X}_2 + \underline{B}_1\underline{u}$$

$$= \underline{A}_{11}\underline{X}_1 + \underline{A}_{12}\left[\underline{V}_{21}\underline{V}_{11}^{-1}\underline{X}_1 - (\underline{V}_{22} - \underline{V}_{21}\underline{V}_{11}^{-1}\underline{V}_{12})\underline{\Lambda}_2^{-1}\underline{B}_2^{*}\underline{u}\right] + \underline{B}_1\underline{u}$$

$$= (\underline{A}_{11} + \underline{A}_{12}\underline{V}_{21}\underline{V}_{11}^{-1})\underline{\tilde{X}} + (\underline{B}_1 - \underline{A}_{12}\underbrace{(\underline{V}_{22} - \underline{V}_{21}\underline{V}_{11}^{-1}\underline{V}_{12})\underline{\Lambda}_2^{-1}\underline{B}_2^{*}}_{B_2'})\,\underline{u}$$

Remarks :

It is called modal approach, because the eigenvalues of the reduced system are a subset of the eigenvalues of the higher order system.

Question: Exist the inverse matrices?

$\underline{V}_{11}$; $\underline{\Lambda}_2$

$$\underline{V} = \begin{bmatrix} \underline{V}_{11} & \underline{V}_{12} \\ \underline{V}_{21} & \underline{V}_{22} \end{bmatrix}$$ matrix of eigenvectors

all the eigenvectors are linearly independend, $\underline{V}$ is regular. That does not mean that $\underline{V}_{11}$ is regular!

e.g. $$\underline{V} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{\Lambda} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}$$

Consequence: By rearranging eigenvectors a regular $\underline{V}_{11}$ can be achieved, but not all dominant Eigenvalues are in the reduced order modal!

$\underline{X} = \underline{V}\,\underline{z}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \cdots \\ x_2 \end{bmatrix}$$

Another possibility to get a regular $\underline{V}_{11}$ is to exchange state variables of $\underline{x}_1$ with those of $\underline{x}_2$ → probably not all output variables can be calculated.

— what is a dominant Eigenvalue?

- unstable eigenvalues are allways "dominant"
- stable eigenvalues are dominant when they are near the imaginary axis.

Example: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 0 \\ 1 & -2 \end{bmatrix}}_{\underline{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$ high order system

Eigen values $\lambda_1 = -5$, $\lambda_2 = -2$ dominant $\lambda_2 = -2$

$y = \underbrace{[\,1 \;\vdots\; 0\,]}_{\underline{c}^T} \underline{x}$ $\qquad y = x_1$

Original system $\dot{x}_1 = -5x_1 + u$ $\rightarrow$ x_1 can be calculated decomposed of x_2!

$\dot{x}_2 = x_1 - 2x_2$

$y = x_1$

The dynamics of x_1 is dominated by $\lambda_1 = -5$.

If we reduce $\lambda_2 = -2$ as dominant Eigenvalue in the reduced order model, consequence would be bad approximation.

Result: Dominance of an eigenvalue is not determined by the location, but also by the influence the output variable which is of interest.

Exchange of state variables:

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_j \\ x_n \end{bmatrix} \qquad \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \qquad y = \underline{c}\,\underline{x}$$

; exchange $x_i \longleftrightarrow x_j$

$$\hat{\underline{x}} = \begin{bmatrix} \hat{x}_1 \\ --- \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ x_m \\ x_{m+1} \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$\dot{\hat{\underline{x}}} = \hat{\underline{A}}\,\hat{\underline{x}} + \hat{\underline{B}}\,u$$

$$\underline{y} = \hat{\underline{c}}\,\hat{\underline{x}}$$

for $\hat{\underline{c}}$.

$$\underline{y} = \underline{c}\,\underline{x} = \begin{array}{c} \begin{array}{ccc} i & & j \end{array} \\ \begin{bmatrix} \begin{bmatrix} \underline{c}_i \end{bmatrix} & \begin{bmatrix} \underline{c}_j \end{bmatrix} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_j \\ x_n \end{bmatrix}$$

$$\underline{y} = \hat{\underline{c}}\,\hat{\underline{x}} \qquad \begin{bmatrix} \begin{bmatrix} \underline{c}_j \end{bmatrix} & \begin{bmatrix} \underline{c}_i \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_m \\ x_{m+1} \\ x_i \\ x_n \end{bmatrix}$$

for $\hat{\underline{B}}$ $\quad \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_i \\ \vdots \\ \dot{x}_m \\ \dot{x}_{m+1} \\ \dot{x}_j \\ \dot{x}_n \end{bmatrix} = \underline{A}\,\underline{x} + \begin{array}{c} \begin{bmatrix} [\underline{b}_i]^T \\ \\ [\underline{b}_j]^T \end{bmatrix} \begin{array}{l} \leftarrow i \\ \\ \leftarrow j \end{array} \end{array} \underline{u} \qquad \dot{\hat{\underline{x}}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_j \\ \vdots \\ \dot{x}_{m+1} \\ \dot{x}_i \\ \vdots \\ \dot{x}_n \end{bmatrix} = \hat{\underline{A}}\,\hat{\underline{x}} + \begin{bmatrix} [\underline{b}_j^T] \\ \\ [\underline{b}_i^T] \end{bmatrix} \underline{u}$$

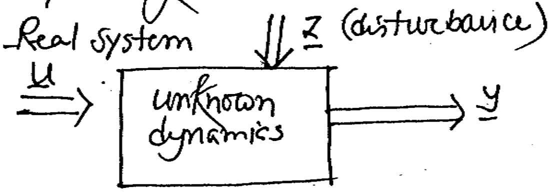
for $\hat{\underline{A}}$ exchange both rows and colums.

Result: Exchange of state variables:
- exchange the according columns in $\underline{C}$
- the rows in $\underline{B}$
- the according rows & columns in $\underline{A}$

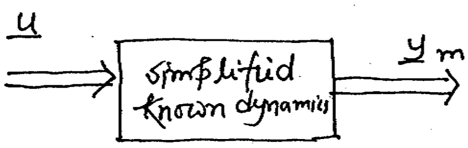
4. Identification

- theoretical part: modelling etc.
- Practical Analysis: Trying to find as simple as possible model, which approximate as good as possible the dynamics of the real system by using test data.

Real system

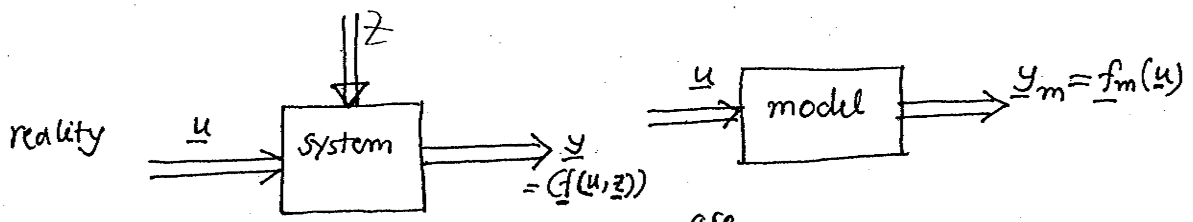


model



parameter adaption by optimizing the approximation!

Identification:

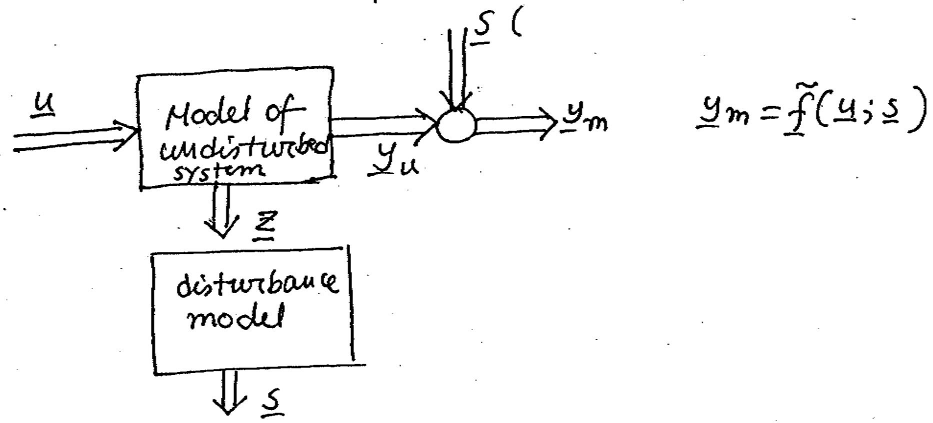


structure and/or parameters of the system are partly ~~+~~ unknown.

input-output behaviour comparison:

f_m contents the effects of the system and disturbance!

Therefore: Model structure (Picture)
fictive



Open tasks.

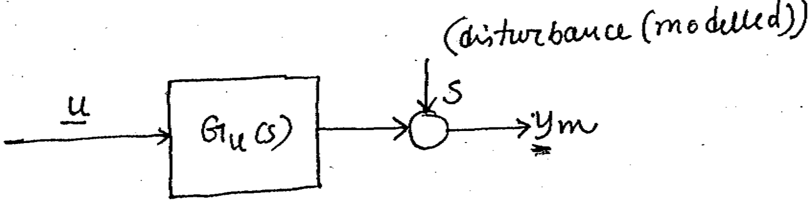
- Find structure and parameters of the undisturbed system!
- find an appropriate disturbance model!

$\Rightarrow y_m \simeq y$

The input u must excite the dominant dynamics to get information about structure and parameters out of the system!

Input ~~system~~ signal:

linear system (assumption)
time domain and frequency domain.

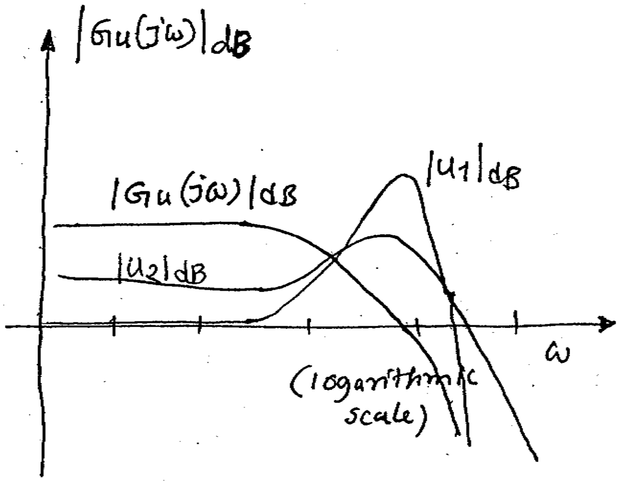


Transfer function $G_u(s) \xrightarrow[s=j\omega]{}$ frequency function $G_u(j\omega)$

$$\frac{Y_u(j\omega)}{U(j\omega)} = G_u(j\omega) = \underbrace{|G_u(j\omega)|}_{\substack{\text{absolute}\\\text{value}\\\text{(amplication)}}} e^{j\angle G_u(j\omega)}$$

Phase of $G_u(j\omega)$

Graphical representation of Bode-Diagram



$$|G_u(j\omega)|_{dB} = 20\log|G_u(j\omega)|$$

$u(j\omega) \to$ Fourier trans.

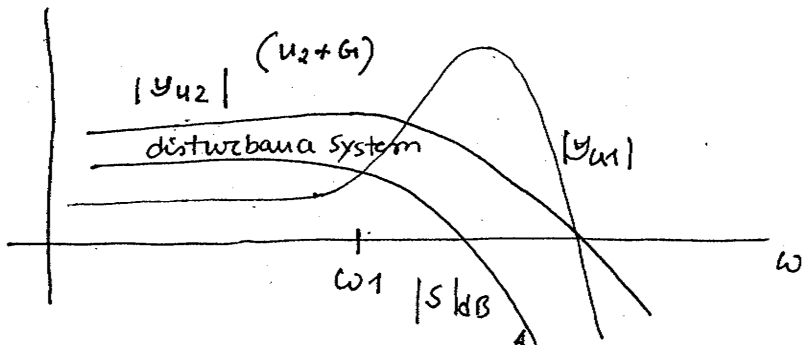
$$= |u(j\omega)| \cdot e^{j\angle u(j\omega)}$$

$$y_u = G_u(j\omega) \cdot u(j\omega)$$

$$|y_u|_{dB} = 20\log\left(|G_u(j\omega)|\,|u(j\omega)|\right)$$

$$= 20\log(|G_u(j\omega)|) + 20\log(|u(j\omega)|)$$

$$= |G_u|_{dB} + |U|_{dB}$$



To get most information about G_u:

- for $\omega < \omega_1$: u_2 delivers a higher response than disturbance $s \to$ more information about G_u
- for $\omega > \omega_1$: u_1 delivers a higher response $\to$ more information about G_u.

$\Rightarrow$ Use that input signal which provides the ~~best~~ best relation $\frac{|y_u|_{dB}}{|S|_{dB}}$ depending on the interested frequency rance.

classification of Input signals:

- natural inputs, indirect access, that means the input of the system is output of another system; and only output of the other system can be influenced.
- artificial inputs, direct access.
  - stochastic signals and deterministic signals.

difficult to generate

easy to generate

Order Reduction

$$u \longrightarrow \boxed{\frac{1}{s+1}} \xrightarrow{x_2} \boxed{\frac{1}{s+2}} \xrightarrow{x_3} \boxed{\frac{1}{s+5}} \xrightarrow{x_1} y$$

$$x_1(s) = \frac{1}{s+5} x_3(s) \Rightarrow sX_1 = -5X_1 + X_3 \quad \bullet\!\!-\!\!\circ \quad \dot{x}_1 = -5x_1 + x_3$$

$$x_3(s) = \frac{1}{s+2} x_2(s) \Rightarrow sX_3 = -2X_3 + X_2 \quad \bullet\!\!-\!\!\circ \quad \dot{x}_3 = -2x_3 + x_2$$

$$x_2(s) = \frac{1}{s+1} u \Rightarrow sX_2 = -X_2 + u \quad \bullet\!\!-\!\!\circ \quad \dot{x}_2 = -x_2 + u$$

$y = x_1$

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \hline \dot{x}_3 \end{bmatrix} = \overset{A_{11} \qquad A_{12}}{\begin{bmatrix} -5 & 0 & \vline & 1 \\ 0 & -1 & \vline & 0 \\ \hline 0 & 1 & \vline & -2 \end{bmatrix}}_{A_{21} \qquad A_{22}} \overset{\underline{x}_1}{\begin{bmatrix} x_1 \\ x_2 \\ \hline x_3 \end{bmatrix}}_{\underline{x}_2} + \overset{B_1}{\begin{bmatrix} 0 \\ 1 \\ \hline 0 \end{bmatrix}}_{B_2} u$$

Reduce 3rd to 2nd order

$$y = [\underbrace{1 \quad 0}_{C_1} \;\vert\; 0]\, \underline{x}$$

Calculate Eigen values:

$$\det(s\underline{I} - \underline{A}) = 0$$

$$= \begin{vmatrix} s+5 & 0 & -1 \\ 0 & s+1 & 0 \\ 0 & -1 & s+2 \end{vmatrix} = 0 \Rightarrow s = -5, -1, -2$$

take out the smallest one, less dominant

$$\underline{\Lambda} = \begin{bmatrix} -1 & 0 & \vline & 0 \\ 0 & -2 & \vline & 0 \\ \hline 0 & 0 & \vline & \underbrace{-5}_{\underline{\Lambda}_2} \end{bmatrix}$$

calculate Eigenvalue Matrix $\underline{V}$

$$\underline{V}\underline{\Lambda} = \underline{A}\,\underline{V} \Rightarrow \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -v_{11} & -2v_{12} & -5v_{13} \\ -v_{21} & -2v_{22} & -5v_{23} \\ -v_{31} & -2v_{32} & -5v_{33} \end{bmatrix} = \begin{bmatrix} -5v_{11}+v_{31} & -5v_{12}+v_{32} & -5v_{13}+v_{33} \\ -v_{21} & -v_{22} & -v_{23} \\ v_{21}-2v_{31} & v_{22}-2v_{32} & v_{23}-2v_{33} \end{bmatrix}$$

$$\left.\begin{array}{ll} \Rightarrow -v_{11} = -5v_{11}+v_{31} & \Rightarrow v_{31} = 4v_{11} \\ \bullet\ -2v_{12} = -5v_{12}+v_{32} & \Rightarrow v_{32} = 3v_{12} \\ \bullet\ -5v_{13} = -5v_{13}+v_{33} & \Rightarrow v_{33} = 0 \end{array}\right\} \text{1 row}$$

$$\left.\begin{array}{ll} -v_{21} = -v_{21} & \Rightarrow \text{choose } v_{21} \\ -2v_{22} = -v_{22} & \Rightarrow v_{22} = 0 \\ -5v_{23} = -v_{23} & \Rightarrow v_{33} = 0 \end{array}\right\} \text{2 row}$$

$$\left.\begin{array}{ll} \bullet\ -v_{31} = v_{21}-2v_{31} & \Rightarrow v_{31} = v_{21} \\ -2v_{32} = v_{22}-2v_{32} & \Rightarrow v_{22} = 0 \\ -5v_{33} = v_{23}-2v_{33} & \Rightarrow 3v_{33} = v_{33} = 0 \end{array}\right\} \text{3 row}$$

$$\underline{V} = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad\qquad}^{\underline{V}_{11}} & \underline{V}_{12} \end{array} \\ \left[\begin{array}{cc|c} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ \hline v_{31} & v_{32} & v_{33} \end{array}\right] \\ \begin{array}{cc} \underbrace{\qquad\qquad\qquad}_{\underline{V}_{21}} & \underline{V}_{22} \end{array} \end{array}$$

Now choose $v_{21} = 4$

$v_{31} = 4$

$v_{11} = 1$

choze $v_{12} = 1$

$\hookrightarrow v_{32} = 3$

choose $v_{13} = 1$

$$\underline{V} = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 0 & 0 \\ \hline 4 & 3 & 0 \end{array}\right]$$

$$\underline{V}^{-1} = \frac{1}{\det \underline{V}} \operatorname{adj} \underline{V}^T = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{12} & -\frac{1}{3} \end{bmatrix}$$

$$\underline{V}_{11}^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & -\frac{1}{4} \end{bmatrix}$$

$$\underline{\Lambda}_2^{-1} = (-5)^{-1} = -\frac{1}{5}$$

$\underline{\Lambda}_2^{-1} = \frac{1}{5}$

calculation of $\underline{\tilde{A}}, \underline{\tilde{B}}$

$$\underline{B}^* = \underline{V}^{-1} \cdot \underline{B} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 1 & \frac{1}{12} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{3} \\ \hline \frac{1}{12} \end{bmatrix}$$

$\underline{B}_2^*$ $\qquad \underline{B}_2^* = \frac{1}{12}$

$$\underline{\tilde{G}}_2 = -\underline{\Lambda}_2^{-1} \underline{B}_2^* = -\left(-\frac{1}{5}\right) \cdot \frac{1}{12} = \frac{1}{60}$$

$$\underline{B}_2' = \underbrace{(\underline{V}_{22} - \underline{V}_{21} \underline{V}_{11}^{-1} \underline{V}_{12}) \underline{\Lambda}_2^{-1} \underline{B}_2^*}_{-G_2}$$

$$= \left(0 - \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \cdot \left(-\frac{1}{60}\right) = +\frac{1}{20} \qquad +\frac{1}{20}$$

$$\underline{\tilde{B}} = \underline{B}_1 \cancel{-} \underline{A}_{12} \cdot \underline{B}_2' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cancel{-} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \cancel{-}\frac{1}{20} = \begin{bmatrix} \cancel{-}\frac{1}{20} \\ 1 \end{bmatrix}$$

$$\underline{\tilde{A}} = \underline{A}_{11} + \underline{A}_{12} \underline{V}_{21} \underline{V}_{11}^{-1} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & -\frac{1}{4} \end{bmatrix}$$

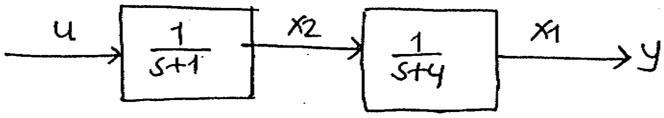
$$\underline{\tilde{A}} = \begin{bmatrix} -2 & \frac{1}{4} \\ 0 & -1 \end{bmatrix}$$

$$\dot{\underline{\tilde{x}}} = \underline{\tilde{A}} \cdot \underline{\tilde{x}} + \underline{\tilde{B}} \underline{u} = \begin{bmatrix} -2 & \frac{1}{4} \\ 0 & -1 \end{bmatrix} \underline{\tilde{x}} + \begin{bmatrix} -\frac{1}{20} \\ 1 \end{bmatrix} \underline{u}$$

$$y = \underline{c}_1 \underline{\tilde{x}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{\tilde{x}}$$

Hoorray! 2nd order system.

Exercise:



$$X_1(s) = \frac{1}{s+4} X_2(s) \Rightarrow sX_1 + 4X_1 = X_2 \Rightarrow sX_1 = -4X_1 + X_2$$

$$\dot{x}_1 = -4x_1 + x_2$$

$$X_2(s) = \frac{1}{s+1} u \quad \Rightarrow sX_2 + X_2 = u \quad \Rightarrow sX_2 = -X_2 + u$$

$$\Rightarrow \dot{x}_2 = -x_2 + u$$

$y = x_1 = [$

$\Rightarrow y = x_1 \dots$

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{array}{c} \underline{A}_{11} \quad \underline{A}_{12} \\ \left[\begin{array}{c|c} -4 & 1 \\ \hline 0 & -1 \end{array}\right] \\ \underline{A}_{21} \quad \underline{A}_{22} \end{array} \underline{x} + \begin{array}{c} \underline{B}_1 \\ \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] \\ \underline{B}_2 \end{array} \underline{u} \qquad y = [1 \quad 0]\, \underline{x}$$

Eigen values : $s = -1, -4$

take out it.

$$\underline{\Lambda} = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & -4 \end{array}\right]$$

$\underline{\Lambda}_2$

calculate Eigenvalue Matrix $\underline{V}$

$$\underline{V}\,\underline{\Lambda} = \underline{A}\,\underline{V}$$

$$\Rightarrow \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

~~First row~~:

$$\Rightarrow \begin{bmatrix} -v_{11} & -4v_{12} \\ -v_{21} & -4v_{22} \end{bmatrix} = \begin{bmatrix} -4v_{11} + v_{21} & -4v_{12} + v_{22} \\ -v_{21} & -v_{22} \end{bmatrix}$$

First row:

$$-v_{11} = -4v_{11} + v_{21} \Rightarrow v_{21} = 3v_{11}$$

$$-4v_{12} = -4v_{12} + v_{22} \Rightarrow v_{22} = 0$$

2nd row :

$-v_{21} = -v_{21}$ $\rightarrow$ choose $v_{21} = 3$

$-4v_{22} = -v_{22}$ $\quad \cancel{v_{22}} = 0$ ok ~~not useful~~

Eigenvalue matrix $\underline{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{array}{cc} \underline{v}_{11} & \underline{v}_{12} \\ \left[\begin{array}{c|c} 1 & 1 \\ \hline 3 & 0 \end{array}\right] & \\ \underline{v}_{21} & \underline{v}_{22} \end{array}$

choose, $v_{12} = 1$

$$\underline{V}^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{bmatrix}$$

$$\underline{\Lambda}_2^{-1} = (-4)^{-1} = -\frac{1}{4}.$$

calculation of $\tilde{\underline{A}}, \tilde{\underline{B}}$

$$\underline{B}^* = \underline{V}^{-1} \cdot \underline{B} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$B_2^* = -\frac{1}{3}$$

$$\tilde{\underline{G}}_2 = -\underline{\Lambda}_2^{-1} \underline{B}_2^* = -\left(-\frac{1}{4}\right) \times \left(-\frac{1}{3}\right)$$

$$= -\frac{1}{12}$$

$$\underline{B}_2' = (\underline{V}_{22} - \underline{V}_{21} \underline{V}_{11}^{-1} \underline{V}_{12}) \underline{\Lambda}_2^{-1} \underline{B}_2^*$$

$$= (0 - 3 \times 1 \times 1) \times -\frac{1}{4} \cdot \times -\frac{1}{3}$$

$$= -\cancel{3} \times \frac{1}{\cancel{12}\,4} = -\frac{1}{4}$$

$$\tilde{\underline{B}} = \underline{B}_1 - \underline{A}_{12} \cdot \underline{B}_2'$$

$$= 0 - 1 \cdot \times -\frac{1}{4} = \frac{1}{4}$$

$$\tilde{\underline{A}} = \underline{A}_{11} + \underline{A}_{12} \underline{V}_{21} \underline{V}_{11}^{-1}.$$

$$= -4 + 1 \times 3 \times 1 = -4 + 3 = -1$$

$$\boxed{\begin{aligned} \dot{\tilde{\underline{x}}} &= \tilde{\underline{A}} \cdot \tilde{\underline{x}} + \tilde{\underline{B}} \cdot \underline{u} = -\tilde{\underline{x}} + \frac{1}{4} u \\ y &= 1 . \tilde{x} = \tilde{x}. \end{aligned}}$$

Lec-18

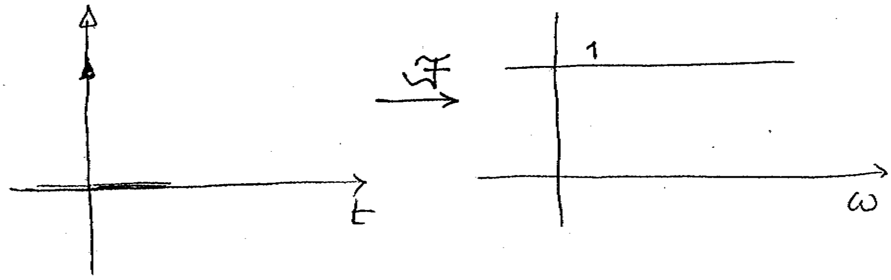
Prof Roth

27/01/2004

Deterministic test signals

- Impulse function

$|u| = A(\omega)$

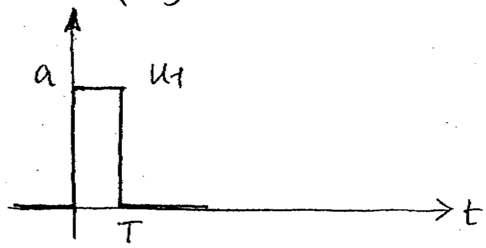


$u(t) = \delta(t)$

Definition of Impulse function: $\int_{-\infty}^{\infty} \delta(t) = 1$, but $\delta(t) = 0$ for $t \neq 0$

$\propto$ small, infinite high $\rightarrow$ technically not reliable

Technical Realizations:



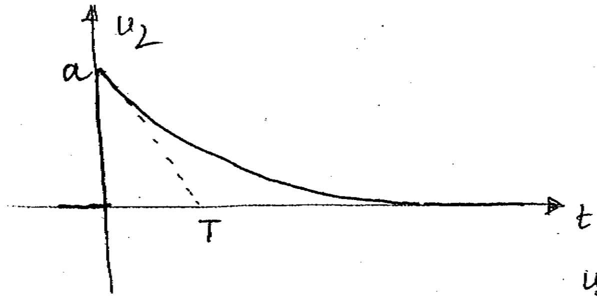
$$u_1(t) = \begin{cases} a & 0 < t < T \\ 0 & t > T \\ 0 & t < 0 \end{cases}$$

$\downarrow \mathcal{F}$

$$A_1(\omega) = aT \frac{\left|\sin \frac{\omega T}{2}\right|}{\frac{\omega T}{2}} \rightarrow si\left(\frac{\omega T}{2}\right)$$

$$= aT \, si \, \frac{\omega T}{2}$$

b)



$$u_2(t) = \begin{cases} 0 & \text{for } t < 0 \\ a e^{-t/T} & \text{for } t > 0 \end{cases}$$

$\downarrow \mathcal{F}$

$$A_2(\omega) = \frac{aT}{\sqrt{1 + (\omega T)^2}}$$

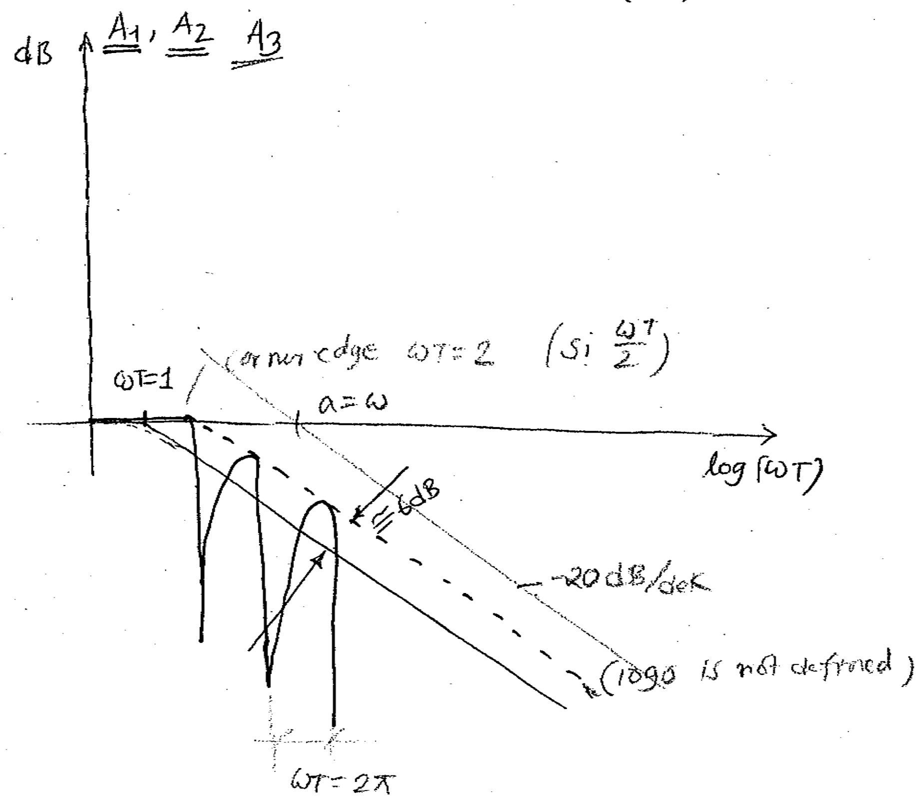
To make the approximations comparable $\int_{-\infty}^{\infty} u_2(t)dt \overset{!}{=} 1$

$$\int_{-\infty}^{\infty} u_1 \, dt \overset{!}{=} 1$$

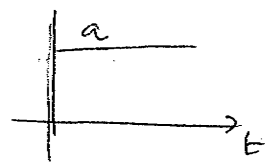
for u_1: $\int_{-\infty}^{\infty} u_1(t)\,dt = aT \stackrel{!}{=} 1$

for u_2: $\int_{-\infty}^{\infty} u_2(t)\,dt = \int_{0}^{\infty} a e^{-t/T}\,dt = -aT e^{-t/T}\Big|_0^{\infty}$

$= -aT(0-1) = aT \stackrel{!}{=} 1$



step function: $u_3(t) = \sigma(t) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t>0 \end{cases}$



$\downarrow \mathcal{F} \longleftarrow u_3(j\omega) = a\pi\delta(\omega) + \frac{a}{j\omega}$

$A_3(\omega) = \frac{a}{\omega}$

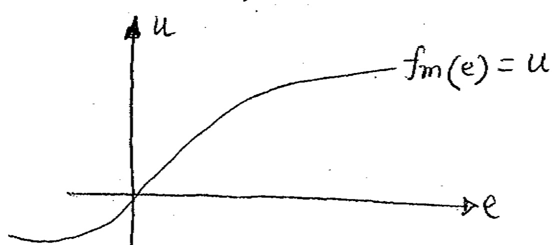
Harmonic functions (sin, cos etc) ; $u_4 = a\sin\omega t \xrightarrow{\mathcal{F}} u_4(j\omega) = a\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$

Classification of the model:

— non parametric models

It exists a functional relation between input and output

- weighting function $g(t)$ o—• $G(s)$ transfer function
- frequency Response function : $G(j\omega)$
- static input-output behavior in non-linear systems,



2n+1 parameters

-- Parametric models

- state space System : $\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}u$
  $y = \underline{c}^T\underline{x} + du$

Task is to find $\underline{A}, \underline{b}, \underline{c}, \underline{d}$

n^2 parameter of $\underline{A}$

n parameter of $\underline{b}$ $\quad n^2+2n+1$

n " " $\underline{c}$

1 " " d

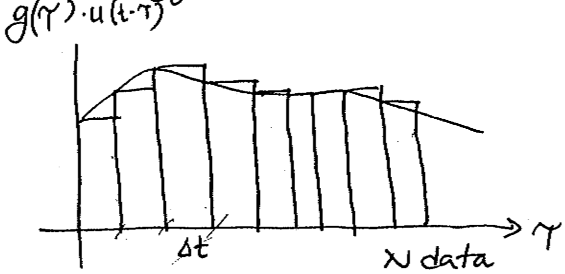
- transfer function $G(s)$: $G(s) = b_0 + b_1 s + b_2 s^2 + \dots b_m s^m$

Mr. Küle

02/02/2004

Non-Parametric Model

$$y(t) = \int_0^t g(\tau) \cdot u(t-\tau)\, d\tau, \quad u(t) = 0 \text{ for } t < 0$$



$g(\tau)$ is unknown, we just have discrete values

Approx:

$$y_m(t) = \sum_{i=0}^{N-1} \underbrace{g(i\cdot\Delta t)\cdot u(t - i\Delta t)}_{\text{height}} \cdot \underbrace{\Delta t}_{\text{width}}$$

Error: $\varepsilon(t) = y(t) - y_m(t)$

$$= y(t) - \sum_{i=0}^{N-1} g(i\cdot\Delta t)\cdot u(t - i\Delta t)\,\Delta t$$

$$\underline{u} = \left[u(N\cdot\Delta t) \quad u((N-1)\cdot\Delta t \ldots \quad u(\Delta t) \right]$$

$$\underline{m}^T = \left[g(0) \quad g(\Delta t) \quad g(2\Delta t) \ldots g((N-1)\Delta t) \right]$$

$\hookrightarrow \varepsilon(t) = y(t) - \underline{u}^T \underline{m} \cdot \Delta t \overset{!}{=} 0$

$y(t) - \underline{u}^T \underline{m} \cdot \Delta t = 0$

$\Rightarrow \underline{m} = y(t) \cdot \underline{u}^{-1} / \Delta t$

so, $\underline{m}$ is measurable

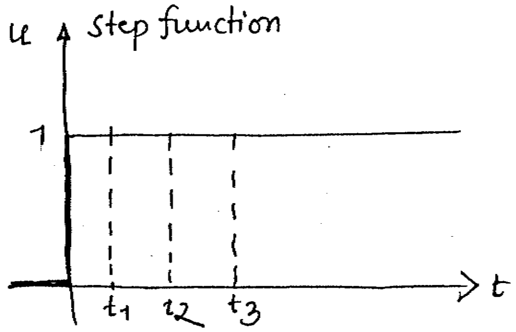
Example: Model $y_m = r_0 + r_1 \cdot u + r_2 \cdot u^2 = \begin{bmatrix} r_0 & r_1 & r_2 \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} 1 & u & u^2 \end{bmatrix}}_{u^T} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}$$

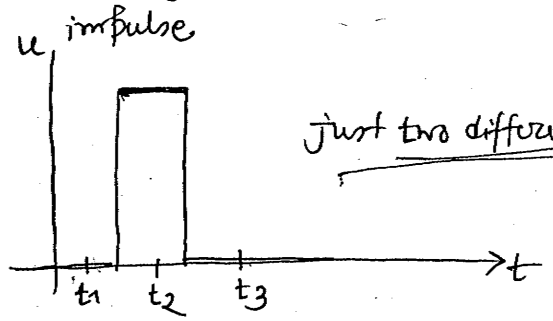
$$\underline{m} = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix} \quad \underline{u} = \begin{bmatrix} 1 \\ u \\ u^2 \end{bmatrix}$$

System $y = \sin(u)$

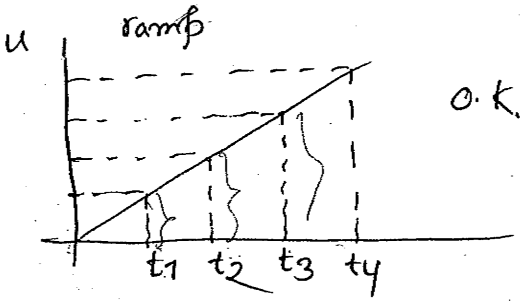
$$\underline{G} = \begin{bmatrix} 1 & u_1 & u_1^2 \\ 1 & u_2 & u_2^2 \\ 1 & u_3 & u_3^2 \end{bmatrix} \qquad \exists\, \underline{G}^{-1} \Rightarrow \det(\underline{G}) \neq 0$$



u always 1, so, G is not invertible



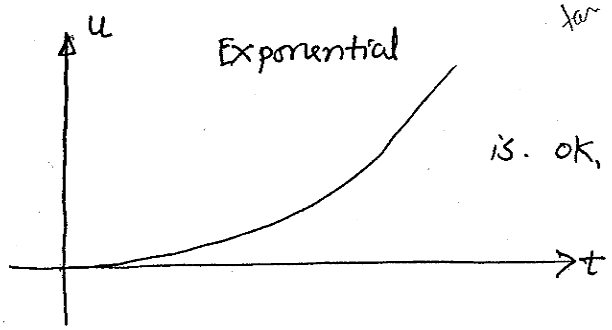
just two different values, ~~G is not invertible~~



O.K.

let's take 45° slope.

$\tan 45^\circ = \frac{u}{t}$



is. ok,

$$\underline{G} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix} \qquad \underline{G}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

$u(t_i)$

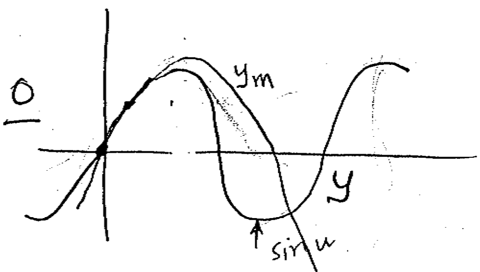
$$\underline{m} = G^{-1} \cdot \underline{y} = \begin{bmatrix} 0 \\ 1.08 \\ -0.24 \end{bmatrix}$$

y is $\sin(u)$ $\rightarrow \underline{y} = \begin{bmatrix} \sin 0 \\ \sin \frac{1}{2} \\ \sin 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.48 \\ 0.84 \end{bmatrix}$

$y_m = \underline{u}^T \cdot \underline{m} = 1.08\, u - 0.24\, u^2$

$\varepsilon = y - y_m = y - \underline{u}^T \underline{m}$

$$\varepsilon_i = \begin{bmatrix} 0 \\ 0.48 \\ 0.84 \end{bmatrix} - \begin{bmatrix} 0 \\ 0.48 \\ 0.84 \end{bmatrix} = \underline{0}$$



Parameters for canonical forms:

control-canonical form:

SISO : $$\underset{(n)}{\dot{\underline{x}}} = \underset{(n,n)}{\begin{bmatrix} 0 & 1 & & & 0 \\ 0 & & 1 & & \\ \vdots & & & 1 & \\ & & & & 1 \\ 0 & & & & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix}} \underline{x} + \underset{(n,1)}{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix}} u$$

$$y = \underset{(1,n)}{\begin{bmatrix} c_1 \cdots & \cdots & c_n \end{bmatrix}} \underline{x} + \underset{(1,1)}{d}\, u$$

$2n+1$ unknown parameters !

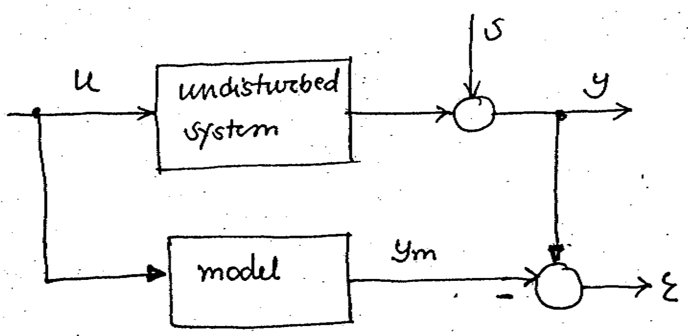
-Z tranfer function: $G_Z(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$

$2n+1$ unknown parameters !

How to find the parameters ?

<u>Linear modelling error</u>

(linearly depending on the unknown parameters)



$\varepsilon = y - y_m = y - y_m(u, \underline{m})$

measurable

$\underline{m}$: parameter vector of the unknown system parameters.

1. Example non-linear static function

e.g. $y_m = r_0 + r_1 u + r_2 u^2 = \underline{u}^T \underline{m}$ $\qquad \underline{u}^T = [1 \quad u \quad u^2]$

r_0, r_1, r_2 → unknown, but u known.

$\underline{\varepsilon} = \underline{y} - \underline{u}^T \underline{m}$ 3 unknowns → at least 3 measurements at different points.

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\underline{y}} - \underbrace{\begin{bmatrix} 1 & u_1 & u_1^2 \\ 1 & u_2 & u_2^2 \\ 1 & u_3 & u_3^2 \end{bmatrix}}_{\underline{G}} \underline{m}$$

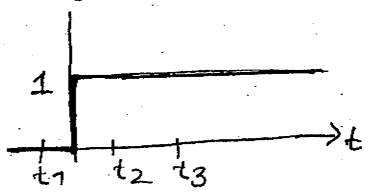
$$\underline{\varepsilon} \stackrel{!}{=} \underline{0}$$

$$\therefore \; \underline{y} = \underline{G}\,\underline{m}$$

if $\underline{G}$ is a quadratic and regular: $\underline{m} = \underline{G}^{-1}\underline{y}$

How can invertibility of $\underline{G}$ be influenced?

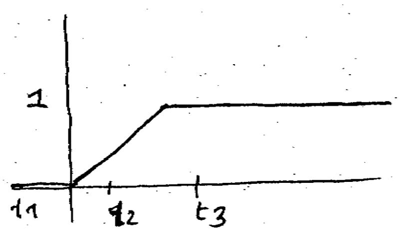
step input:



$$\underline{G} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ both rows are identical, so, G can not be inverted!

$\underline{G}$: singular!

ramp input:

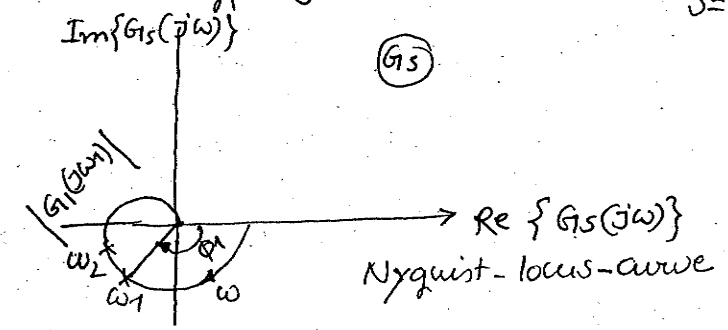


$$\underline{G} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det \underline{G} = 1\,(0.5 - 0.25) = 0.25.$$

$\therefore$ $\underline{G}$ is invertible!!

2. Example: Frequency Response : $G_3(s) \xrightarrow[s=j\omega]{} G_s(j\omega)$



Measure Nyquist-locus-curve
— applying a harmonic signal with frequency ω_1 (u)

Measure Nyquist-locus-curve
- applying a harmonic signal with frequency $\omega_1(u)$, amplitude E
- linear system
- output signal : harmonic signal with ω_1, different Amplitude A; phase shift ϕ_1.

$$|G_S(j\omega_1)| = \frac{A_1}{E_1} \qquad \angle G_S(j\omega_1) = \phi_1$$

2 Equations $\longrightarrow$ 2 parameters can be found $\rightarrow$ 2 unknows in the system structure;

$$G_S(s) = \frac{K}{1+sT} \quad PT1 \;\; (K, T \text{ unknown})$$

$$\xrightarrow{s=j\omega} = \frac{K}{1+j\omega T}$$

Identification means: to find K, T. K, T are positive

$$(1) \qquad |G(j\omega_1)| = \frac{K}{\sqrt{1+(\omega_1 T)^2}}$$

$$(2) \qquad \angle G_2(sj\omega_1) = -\arctan \omega T$$

$$\underline{0} \overset{!}{=} \underline{\varepsilon} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} \frac{K}{\sqrt{1+\omega_1 T}} \\ -\arctan \omega_1 T \end{bmatrix} \qquad \underline{m} = \begin{bmatrix} K \\ T \end{bmatrix}$$

unfortunately model is non-linear depending on $\underline{m}$

usually numerical solution.

Even in cases, where the modelling error is linearly depending on the parameters additional measurements increase the number of equations, but do not improve the quality of the solution:
You get zero error at certain points, but in between ~~you~~ you cannot say anything about the approximation.

Improvement: l unknown parameters; p measurables
$p \gg l$.

$$\varepsilon_1 = y_1 - \underline{u}_1^T \underline{m}$$
$$\vdots$$
$$\varepsilon_l = y_l - \underline{u}_l^T \underline{m}$$
$$\varepsilon_{l+1} = y_{l+1} - \underline{u}_{l+1}^T \underline{m}$$
$$\vdots$$
$$\varepsilon_p = y_p - u_p^T m$$

$$\min_{\underline{m}} \sum_{i=1}^{p} \varepsilon_i^2 = J$$

$$\left.\begin{array}{c} \frac{\partial J}{\partial m_1} \overset{!}{=} 0 \\ \vdots \\ \frac{\partial J}{\partial m_l} \overset{!}{=} 0 \end{array}\right\} \begin{array}{l} l \text{ equations} \\ \text{for } l \text{ unknowns} \end{array}$$