Session 1

2.032
9,8-1

Dynamics

Dynamics: Kinematics and kinetics of particles, rigid bodies and Continua

Kinematics: Studies motion without its Cause

Kinetics: relates forces and torques to motion

Foundations of Dynamics Newton's laws (axioms)

I. The existence of inertial frame

a free particle stays fix or moves uniformely along a line

II. In an inertial frame, "F=ma"

III. Action & Reaction forces are equal & act in opposite directions

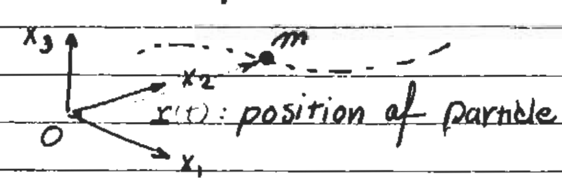
This class applies these laws to
- particles
- Systems of particles
- Rigid bodies / Systems of Rigid bodies

Two approaches: 1) Newton-Euler approach (vectorial) $\Rightarrow$ reaction forces

2) Lagrangian-Hamiltonian approach (scalar) $\Rightarrow$ equations of motion

(I) Newton-Euler mechanics
(Newtonian)

(1) Dynamics of a particle



$[x_1, x_2, x_3]$ is an inertial frame

Velocity. $\underline{v}(t) = \dot{\underline{r}}(t) = \frac{d}{dt}\underline{r}(t)$

$\underline{a}(t) = \dot{\underline{v}}(t) = \ddot{\underline{r}}(t)$

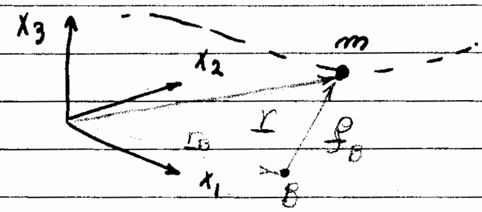
(a) Linear momentum principle

Define: $\underline{P} = m\underline{v}$ linear momentum

Newton II $\Rightarrow \dot{\underline{P}} = \underline{F}$ resultant Force

if $\underline{F} = \underline{0} \Rightarrow \underline{P} = \text{Const}$ Conservation of linear momentum

(b) Angular momentum principle



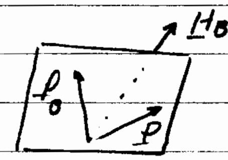
B Can move.

B: a point in $[x_1, x_2, x_3]$ frame (potentially moving)

Define: $\underline{H}_B = \underline{\rho}_B \times \underline{P}$

In words $\underline{H}_B$ is the moment of the lin. momentum w.r.t. B

Also define $\underline{M}_B = \underline{\rho}_B \times \underline{F}$



(resultant torque w.r.t. B)

$$\dot{\underline{H}}_B = \frac{d}{dt}(\underline{\rho}_B \times \underline{P}) = \dot{\underline{\rho}}_B \times \underline{P} + \underline{\rho}_B \times \dot{\underline{P}}$$

$$= (\dot{\underline{r}} - \dot{\underline{r}}_B) \times \underline{P} + \underbrace{\underline{\rho}_B \times \underline{F}}_{\underline{M}_B}$$

$$= -\dot{\underline{r}}_B \times \underline{P} + \underline{M}_B \quad \text{because } (\dot{\underline{r}} \parallel \underline{P})$$

$$\dot{\underline{H}}_B + \underline{v}_B \times \underline{P} = \underline{M}_B$$

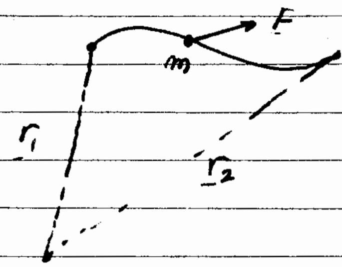
(*) ($\dot{\underline{H}}_B = \underline{M}_B$ if $\underline{v}_B = 0$ or $\underline{v}_B \parallel \underline{P}$ and the second includes the first)

If $\underline{M}_B = \underline{0}$ AND (*) holds then $\underline{H}_B = \text{Const}$

Conservation of angular momentum

(c) Work-Energy Principle

Define $W_{12} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r}$



work done by resultant force

$d\underline{r} = \underline{V}\,dt$

$\underline{F} = m\underline{\dot{V}} \quad \Rightarrow W_{12} = \int_{t_1}^{t_2} m \underline{\dot{V}}\,\underline{V}\,dt$

$$= \int_{t_1}^{t_2} \frac{d}{dt}\Big(\frac{1}{2} m \underbrace{\underline{V}\cdot\underline{V}}_{|V|^2}\Big)\,dt$$

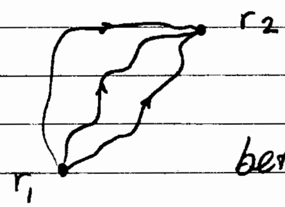
$$= \frac{1}{2} m\,|V_2|^2 - \frac{1}{2} m\,|V_1|^2$$

Define $T = \frac{1}{2} m |V|^2$ Kinetic Energy

$\Rightarrow W_{12} = T_2 - T_1$

(work by F equals to change in kinetic Energy)

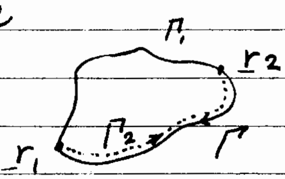
Assume that particle moves in a Force field $\underline{F}(\underline{x},\dot{\underline{x}}) = \underline{F}(\underline{x})$, Such that



$\int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r}$ is independent of the path between $\underline{r}_1$ & $\underline{r}_2$

Then $\underline{F}(x)$ is Called Conservative.

Consequence



Γ: closed curve

$\Gamma = \Gamma_1 \cup (-\Gamma_2)$

(Γ_2 is defined in opposite direction)

$\oint_{\Gamma} \underline{F}\,d\underline{r} = \int_{\Gamma_1} \underline{F}\,d\underline{r} - \int_{\Gamma_2} \underline{F}\,d\underline{r} = \underline{0}$

$\downarrow$

$\underline{F}(x)$ is Conservative

E.g. field of gravity is a Conservative fields

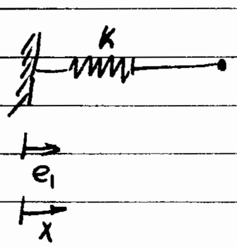
By potential theory, for a Conservative $\underline{F}(x)$, There exists $V(\underline{x})$ (the potential) Such that $\underline{F} = -\nabla V$ (- grad V)

$$= -\left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}\right)$$

Eg. gravitational field

$g\downarrow$ $\downarrow y$ $\overset{m}{\downarrow} mg$ $\quad \underline{F} = m\underline{g} \Rightarrow V = mgy$

Eg. Spring force



$\underline{F} = -kx\underline{e}_1 \Rightarrow$ potential $V = \frac{1}{2}kx^2$

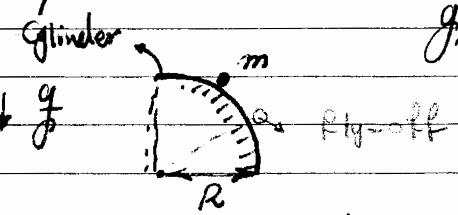
$$\Rightarrow \quad W_{12} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\, d\underline{r} = \int_{\underline{r}_1}^{\underline{r}_2} (-\nabla V)\, d\underline{r} = V_1 - V_2$$

$$\Rightarrow T_2 - T_1 = V_1 - V_2$$

$$\Rightarrow T_1 + V_1 = T_2 + V_2$$

$\Rightarrow E = T + V$ total mechanical Energy in Conserved in a potential force field

Example: point mass slides on cylinder under the effect of gravity.



- what is the fly-off angle?
- How does φ^* depend on R & m?

Session 2

2_032
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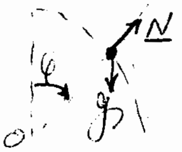
Example (1)

$\downarrow g$ 

Question: angle at departure
Point of departure is the point where the reaction force acting on particle becomes zero.

FBD 

Linear momentum principle $\quad \dot{p} = F$

From geometry $\quad \underline{r} = (\cos\varphi\, \underline{e}_1 + \sin\varphi\, \underline{e}_2) R$

$$\underline{v} = \dot{\underline{r}} = (-\sin\varphi\,\dot\varphi\, \underline{e}_1 + \cos\varphi\,\dot\varphi\, \underline{e}_2) R$$

$$\underline{a} = \ddot{\underline{r}} = \{(-\cos\varphi\,\dot\varphi^2 - \sin\varphi\,\ddot\varphi)\, \underline{e}_1 + (-\sin\varphi\,\dot\varphi^2 + \cos\varphi\,\ddot\varphi)\, \underline{e}_2\} R$$

$$\Rightarrow \dot{p} = m\dot{\underline{v}} = m\underline{a} = mR(-\cos\varphi\,\dot\varphi^2 - \sin\varphi\,\ddot\varphi)\, \underline{e}_1 + mR(-\sin\varphi\,\dot\varphi^2 + \cos\varphi\,\ddot\varphi)\, \underline{e}_2$$

$$= \underline{N} + m\underline{g}$$

Project in the direction of $\underline{e}_N$
multiply by $\quad \underline{e}_N = \cos\varphi\, \underline{e}_1 + \sin\varphi\, \underline{e}_2$

$$-mR\dot\varphi^2 = N - mg\sin\varphi$$

At the point of departure $N = 0$

$$\Rightarrow R\dot\varphi_*^2 = g\sin\varphi_*$$

To obtain another equation for $(\varphi_*, \dot\varphi_*)$, use work-Energy principle

$$W_{12} = T_2 - T_1$$

$$W_{12} = \int_1^2 \underline{F}\cdot d\underline{r} = \int_1^2 \cancel{\underline{N}\, d\underline{r}}^{\,0} + \int_1^2 m\underline{g}\, d\underline{r}$$

$$\underline{F} = -\nabla V$$

Since N doesn't do any work ($\underline{N} \perp \underline{v}$) $\Rightarrow$ System is Conservative $\quad T + V = \text{Const}$

$$0 + mgR = \frac{1}{2} m|\underline{v}|^2 + mgR\sin\varphi$$

$$|\underline{v}| = \left|\frac{d}{dt}\left(R\left(\frac{\pi}{2} - \theta\right)\right)\right| = R^2\dot\varphi^2$$

$$\Rightarrow \frac{1}{2} R^2\dot\varphi_*^2 = gR(1 - \sin\varphi_*) \quad \xrightarrow{\text{both}} \sin\varphi^* = \frac{2}{3} \Rightarrow \varphi^* \doteq 41.81^\circ$$

Example (2) ($\approx$ Dynamics qual exam, Spring 2004)

$g\downarrow$ 

upon Collison rod pivots at point B
Question: minimum v for which rod tips over

Part 1: Collision itself $(t_- \to t_+)$

Forces at B are unknown $\Rightarrow$ use angular momentum principle w.r.t B

$$\dot{H}_B + \cancel{\underline{V}_B \times \underline{P}}^{\,0} = M_B = \underline{r}_{BC} \times (m\underline{g})$$

Integrate from t_- to t_+

$$\underline{H}_B(t^+) - H_B(t^-) = \int_{t_-}^{t_+} \underline{r}_{BC} \times (m\underline{g})\, dt \xrightarrow{t_- \to t_+} 0$$

$$\Rightarrow \underline{H}_B(t_-) \xrightarrow{t_- \to t_+} \underline{H}_B(t_+)$$

$\Rightarrow \underline{H}_B$ is Conserved during Collision

$$H_B(t_-) = \underline{r}_{BC} \times \underline{P}(t_-) = \underline{r}_{BC} \times (m\underline{V}_-) = \left(\frac{2L}{3} \, \mathcal{E}\alpha\right) mV_- \, \underline{e}_3$$

$$H_B(t_+) = \underline{r}_{BC} \times \underline{P}(t_+) = \underline{r}_{BC} \times (m\underline{V}_+) = \frac{2L}{3} m V_+ \, \underline{e}_3$$

$$\Rightarrow V_+ = V_- \, \mathcal{E}\alpha$$

Part 2: Rotation

work-Energy principle $\quad W_{12} = T_2 - T_1$

$$W_{12} = W_{12}^{gravity} + \underbrace{W_{12}^{reaction}}_{0}$$

(B does not move)

$\Rightarrow$ motion is Conservative $\qquad W_{12} = V_1 - V_2$

$$T + V = Const$$

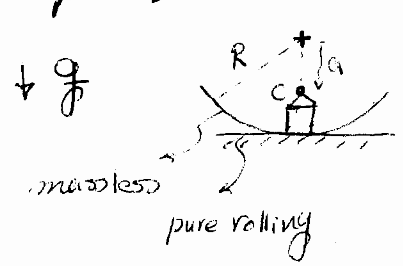
critical case

$$\frac{1}{2} \cancel{m} V^2 + \cancel{m} g \frac{2L}{3} \sin\alpha = 0 + \cancel{m} g \frac{2L}{3}$$

$$V_+^2 = \frac{4L}{3} g (1 - \mathcal{E}\alpha)$$

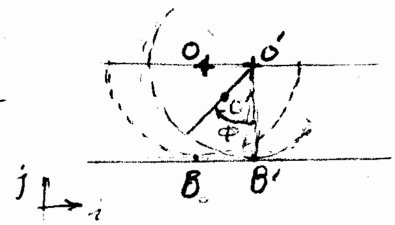
$$V_- = \sqrt{\frac{4Lg}{3}\left(\frac{1}{\sin\alpha} - \frac{1}{\mathcal{E}\alpha}\right)}$$

Example (3)

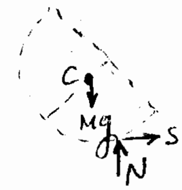


Question: equations of motion
- Constraint force
- frequency of small Oscillations

φ: Generalized Coordinates
(Complete & independent Set of Coordinates)



Start with F.B.D.



rolling means there is no relative motion between two Surfaces involved

Use angular momentum principle Start at B'

$$\dot{H}_B + \underline{V}_B \times \underline{P} = M_B$$

$$\underline{H}_B = \underline{r}_{BC} \times \underline{P} = \underline{r}_{BC} \times m\underline{V}_C$$

$$\underline{r}_C = (R\varphi - a\sin\varphi)\,\underline{i} + (R - a\cos\varphi)\,\underline{j}$$

$$\underline{V}_C = \dot{\underline{r}}_C = (R - a\cos\varphi)\dot{\varphi}\,\hat{i} + a\sin\varphi\,\dot{\varphi}\,\underline{j}$$

$$\underline{r}_{BC} = -a\sin\varphi\,\underline{i} + (R - a\cos\varphi)\,\underline{j}$$

$$\Rightarrow \underline{H}_B = m\left[-a^2 - R^2 + 2aR\cos\varphi\right]\dot{\varphi}\,\underline{k}$$

$$\dot{\underline{H}}_B = m\left[(2ar\cos\varphi - a^2 - R^2)\ddot{\varphi} - 2aR\sin\varphi\,\dot{\varphi}^2\right]\underline{k}$$

$$\underline{V}_B = \frac{d}{dt}(R\varphi)\,\hat{i} = R\dot{\varphi}\,\hat{i}$$

$$\underline{P} = m\underline{V}_C \qquad V_B \times \underline{P} = a\,R\sin\varphi\,m\dot{\varphi}^2\,\underline{k} = maR\sin\varphi\,\dot{\varphi}^2\,\underline{k}$$

$$\overline{M}_B = mga\sin\varphi\,\underline{k}$$

we get

$$(R^2 + a^2 - 2aR\cos\varphi)\ddot{\varphi} + aR\sin\varphi\,\dot{\varphi}^2 + ag\sin\varphi = 0$$

eq of motion
2nd order ODE, nonlinear
2nd order ODE
initial Conditions $\varphi(t_0) = \varphi_0$, $\dot{\varphi}(t_0) = \dot{\varphi}_0$

For numerical Solutions, let $x_1 = \varphi$, $x_2 = \dot{\varphi}$. Eq of motion becomes 1st order System of ODE's

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{aR\sin x_1\,x_2^2 + ag\sin x_1}{R^2 + a^2 - 2aR\cos x_1}$$

$$x_1(0) = \varphi_0$$

$$x_2(0) = \dot{\varphi}(0)$$

To obtain the reaction forces use linear momentum $\dot{\underline{P}} = \underline{F}$

(x) equation $m\ddot{x}_c = S$ $\qquad x_c = R\varphi - a\sin\varphi$

$$\Rightarrow S = m\left[a\sin\varphi\,\dot\varphi^2 + (R - a\cos\varphi)\ddot\varphi\right]$$

use eq of motion $\Rightarrow S = m\left[a\sin\varphi\,\dot\varphi^2 - \dfrac{(R-a\cos\varphi)(aR\sin\varphi\,\dot\varphi^2 + ag\sin\varphi)}{R^2+a^2-2aR\cos\varphi}\right]$

(y) equation $m\ddot{y}_c = N - mg \Rightarrow N = m(\ddot{y}_c + g)$

$y_c = R - a\cos\varphi$

S and N are non-potential forces because their expersion is also dependent on $\dot\varphi$. if it was just dependent on φ it was.

S, N act on the chair at point $\tilde{B}$ a point at the disc instantaneously at B

$$\underline{V}_{\tilde{B}} = \underline{V}^{rel}_{B/O'} + \underline{V}_{O'} \qquad \text{(to be justified)}$$
$$= -R\dot\varphi\,\hat{i} + R\dot\varphi\,\hat{i}$$
$$= 0$$

$$\Rightarrow \int_{t_1}^{t_2} (\underline{N} + \underline{S})\cdot\underline{V}_{\tilde{B}(t)}\,dt \quad \Rightarrow \quad W_{12} = 0 \quad \Rightarrow \quad \text{System is Conservative}$$

$$\Rightarrow T + V = \frac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2) + mgy_c = \text{Const}$$

$\dfrac{d}{dt}(T+V) = 0$ Substituting of x_c and y_c gives the same eq. of motion

this procedure works if the system has one degree of freedom (1 generalized Coordin the system is Conservative

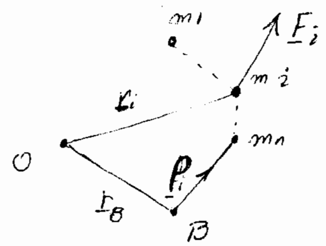
Frequency of Small Oscillations

linearize eq. of motion about $\varphi = 0$, $\dot\varphi = 0$ i.e. taylor expand in two variables φ and $\dot\varphi$ and keep linear terms only

$$\left[R^2 + a^2 - 2aR(1 + \cdots)\right] + aR(\cancel{0 + \cdots})\,\dot\varphi^2 + ag(\varphi + \cdots) = 0$$

$$\underbrace{(R-a)^2}_{\text{mass Coef.}}\ddot\varphi + \underbrace{ag}_{\text{Spring Coef.}}\varphi = 0 \quad \Rightarrow \quad \boxed{\omega = \sqrt{\frac{ag}{(R-a)^2}}}$$

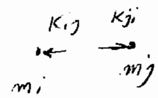
II Dynamics of systems of particles



- A system of n masses
- B: Point in the intertial fram
- F_i: resultant force on m_i

$$\underline{F}_i = \underline{F}_i^{ext} + \underline{F}_i^{int}$$

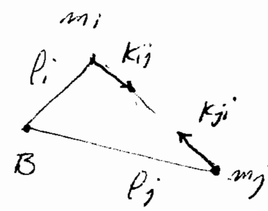
$$F_i^{int} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \underline{K}_{ij}$$



Newton III $\Downarrow$
$K_{ij} + K_{ji} = \underline{0}$ $\Rightarrow$ $\sum_{i=1}^{n} F_i^{int} = \underline{0}$

Also $\sum_{i=1}^{n} \underline{\rho}_i \times \underline{F}_i^{int} = \underline{0}$

Eg. For two masses



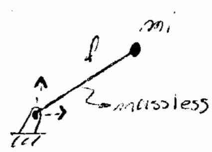
$$\underline{\rho}_i \times \underline{K}_{ij} + \underline{\rho}_j \times K_{ji}$$

$$= (\underline{\rho}_i - \underline{\rho}_j) \times K_{ji} = 0$$

Constraints: geometric limitation on the absolute or relative motion of particle.

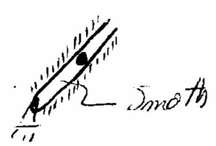
eg. (1)

(3D)



$x_i^2 + y_i^2 + z_i^2 = l^2$

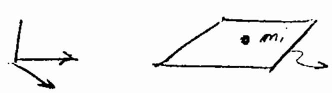
(2)



$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \mu(t)\, \ell(t)$ ↳ prescribed

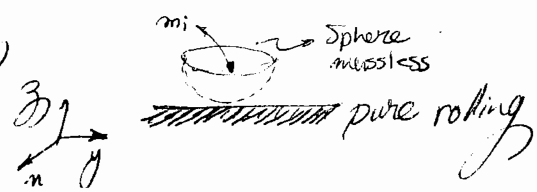
↳ we can eliminate $\mu(t)$ and get 2 Constraints

(3)



$Z_i = 0 \Rightarrow 1$ Constraint

(4)



$Z_i = \text{Const}$

Degrees of freedom

- Degrees of Freedom   Could be time but Not velocity

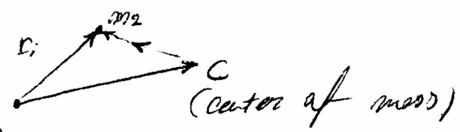
$$\# DOF = \underbrace{3n}_{\text{DOF for unconstraint motion}} - (\# \text{ of indep. Scalar restriction on position})$$

or

$\#$ of Constraints

- Total mass $\quad M = \sum_{i=1}^{n} m_i$

- Center of mass: geometric point w.r.t which the total mass moment is zero



$$\sum_{i=1}^{n} m_i (\underline{r}_i - \underline{r}_c) = 0$$

(a) Linear momentum principle $\quad (*)\ \dot{\underline{P}}_i = \underline{F}_i \ ; \ \underline{P}_i = m_i \dot{\underline{r}}_i$

$$= \underline{F}_i^{ext} + \underline{F}_i^{int}$$

Define: $\underline{P} = \sum_{i=1}^{n} \underline{P}_i = M \frac{1}{M} \sum_i m_i \underline{v}_i =$

$$= M \underbrace{\frac{d}{dt}\underbrace{\left(\frac{1}{M}\sum m_i \underline{r}_i\right)}_{r_c}}_{v_c}$$

$$= M \underline{V}_c$$

Do $\sum$ on $(*)$ $\quad \dot{\underline{P}} = \underline{F}^{ext} \ ; \quad \underline{F}^{ext} = \sum_{i=1}^{n} \underline{F}_i^{ext}$

linear momentum principle

if $\underline{F}^{ext} = \sum_i \underline{F}_i^{ext} = 0 \Rightarrow \underline{P} = \text{Const}$ (Conservation of linear momentum)

b) Angular momentum Principle $\quad \dot{\underline{P}}_i = \underline{F}_i^{ext} + \underline{F}_i^{internal}$

$$\Rightarrow \sum_i \underline{\rho}_i \times \dot{\underline{P}}_i = \frac{d}{dt} \underbrace{\overbrace{\sum \underline{\rho}_i \times \underline{P}_i}^{H_B}}_{\text{def } \underline{H}_B} - \underbrace{\sum_i \dot{\underline{\rho}}_i \times \underline{P}_i}$$

$$= (\dot{\underline{r}}_i - \dot{\underline{r}}_B) \times \underline{P}_i$$

$$= \underbrace{\dot{\underline{r}}_i \times \underline{P}_i}_{0} - \dot{\underline{r}}_B \times \underline{P}_i$$

$$= \dot{\underline{H}}_B + \underline{V}_B \times \underline{P}$$

$$\Rightarrow \dot{\underline{H}}_B + \underline{V}_B \times \underline{P} = \underline{M}_B \qquad \underline{M}_B = \sum_{i=1}^{n} \underline{\rho}_i \times \underline{F}^{ext}$$

If $M_B = 0$
And . B is fixed
. V_B is parallel to $\underline{P}$ $\quad \underline{V_B} \parallel \underline{P}$
or . B is the Center of mass

$\Big\} \Rightarrow H_B = Cte$ Conservation of angular momentum

c) Work-Energy Principle

Have seen: $W_{12}^i = \int_1^2 \underline{F}_i \cdot d\underline{r}_i$

$$T^i = \frac{1}{2} m |\underline{v}_i|^2$$

Define

$$W_{12} = \sum_i W_{12}^i$$

$$T = \sum_i T^i$$

therefor, by $W_{12}^i = T_2^i - T_1^i$

$W_{12} = T_2 - T_1$ Work-Energy principle, includes W_{12}^{int}

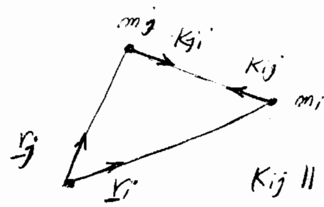
$$W_{12} = \sum_{i=1}^n \int_1^2 \underline{F}_i^{ext} \cdot d\underline{r}_i + \sum_{i=1}^n \underline{F}_i^{int} \cdot d\underline{r}_i \qquad d\underline{r}_i = \underline{v}_i\, dt$$

$$\sum_{i=1}^n \int_{t_1}^{t_2} \underline{F}_i^{int}\, \underline{v}_i\, dt = \int_{t_1}^{t_2} \sum_{\substack{ij \\ i \neq j}} (\underline{K}_{ij}\, \underline{v}_i + \underline{K}_{ji}\, \underline{v}_j)\, dt$$

$$= \int_{t_1}^{t_2} \sum_{\substack{ij \\ i \neq j}} \underline{K}_{ij} \cdot (\underline{v}_i - \underline{v}_j)$$

Important special Case

$\underline{K}_{ij} \cdot (\underline{v}_i - \underline{v}_j) = 0$ for all ij



$Kij \parallel \underline{r}_i - \underline{r}_j$

$$\Rightarrow (\underline{r}_i - \underline{r}_j) \parallel \underbrace{(\underline{v}_i - \underline{v}_j)}_{\frac{d}{dt}(r_i - r_j)}$$

$$\frac{d}{dt}(\underline{r}_i - \underline{r}_j)(r_i - r_j) = 0 \Rightarrow \boxed{\frac{d}{dt}(r_i - r_j)^2 = 0} \text{ for all } ij$$

Definition: Systems of particles with $|r_i - r_j| = Const$ are called rigid-body

For such systems: $W_{12}^{ext} = T_2 - T_1$

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If furthermore all external forces are potential, i.e.,

$$\underline{F}_i^{ext} = -\nabla V_i^i(\underline{r}_j, t)$$

Then

$$T_2 - T_1 = W_{12}^{ext} = \sum_i \int_1^2 \underline{F}_i^{ext} d\underline{r}_i = \sum_i V_1^i - V_2^i = V_1 - V_2$$

Where $V_1 = \sum_i V_1^i$

$V_2 = \sum_i V_i^2$

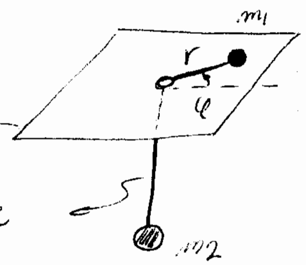
$T + V = \text{Const}$

Conservation of Energy. (rigid body) external forces are potential

Example

Smoth

inextensible



Assume $\dot{r}(0) = 0$

$r(0) = r_0$

$vr \overset{?}{=} c$

$\frac{v^2}{r} = \frac{(\frac{k}{r})^2}{r} = \frac{k^2}{r^3}$

Question: minimum value of r

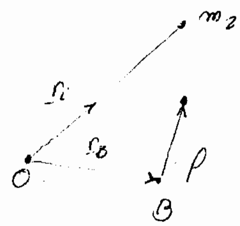
maximum value string force

Dynamics of Systems of particles

(1) $\dot{\underline{P}} = \underline{F}^{(ext)}$

(2) $\dot{\underline{H}}_B + \underline{V}_B \times \underline{P} = \underline{M}_B$

(3) $W_{12} = T_2 - T_1$

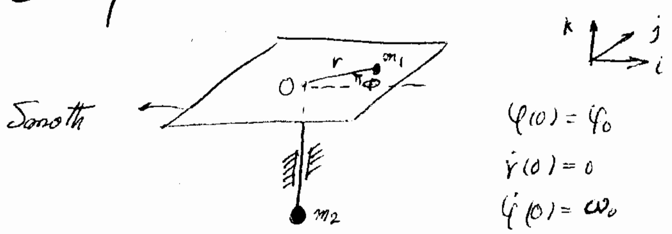


For Rigid body Systems

$$W_{12}^{ext} = 0$$

$T + V = Const$ for Conservative Systems

Example



$\varphi(0) = \varphi_0$

$\dot{r}(0) = 0$

$\dot{\varphi}(0) = \omega_0$

inextensible length l

Questions
- minimal r?
- maximal String force?

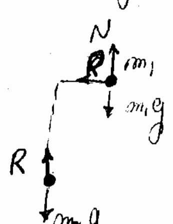
Work-energy principle $\quad W_{12} = T_2 - T_1$

Degrees of freedom: $\#DOF = 2 \times 3 - 1 - 2 - 1$

Plane for m_1 — String — vertical for m_2

Use (r, ϕ) as generalized Coordinate

FBD



- N, $m_1 g$ do not work
- $m_2 g$ is potential
- work done by string forces

$$W_{12}^{int} = \int_1^2 (\underline{R}_1 \cdot d\underline{r}_1 + \underline{R}_2 \cdot d\underline{r}_2) = \int_1^2 R\,dr - \int_1^2 R\,dr = 0$$

$\Rightarrow W_{12} = W_{12}^{ext}$ (all external forces are either potential or don't do work)

$= V_1 - V_2$ (V Potential)

$\Rightarrow \underset{\sim}{T+V} = Const \Rightarrow$ System is Conservative

$$E = \frac{1}{2} m_1 |V_1|^2 + \frac{1}{2} m_2 |V_2|^2 - m g (l - r) = Const \qquad |V_1|^2 = \dot{r}^2 + r^2 \dot{\varphi}^2 \quad |V_2|^2 = \dot{r}^2$$

$$m(\dot{r}^2 + \frac{1}{2} r^2 \dot{\varphi}^2) + m g r = m(\underset{0}{\dot{r}_0^2} + \frac{1}{2} r_0^2 \omega_0^2) + m g r_0$$

Angular momentum principle (w.r.t. O)

$$\dot{\underline{H}}_O + \cancel{\underline{v}_O \times \underline{P}} - \underline{M}_O = 0$$

$\Rightarrow \underline{H}_O = Const$

$$\underline{H}_O = \underline{r}_{om} \times \underline{P}_1 + \cancel{r_{om} \times \underline{P}_2}$$

$$= r m \underbrace{r\dot{\varphi}}_{v_m} \qquad \Rightarrow r_0^2 \omega_0 = r^2 \dot{\varphi}$$

Combine (1) & (2): $\quad m\frac{\omega_0^2 r_0^4}{2r^2} + mgr = \frac{1}{2} m r_0^2 \omega_0^2 + mgr_0$

Cubic eq for r but we know one root at $r=r_0$ we have $\dot{r}=0$

Divide (3) by $r-r_0$ $\Rightarrow r^2 - \frac{\omega_0^2 r_0^2}{2g} r - \frac{\omega_0^2 r_0^3}{2g} = 0$

positive root: $r_{min} = \frac{\omega_0^2 r_0^2}{4g}\left(1+\sqrt{1+\frac{8g}{\omega_0^2 r_0}}\right)$

maximal force in string

linear momentum principle for m_2:

$$\dot{\underline{P}}_2 = \underline{R}_2 - mg$$

$$\Rightarrow m\ddot{r} = R - mg \quad \Rightarrow R = m(\ddot{r} + g) \quad (4)$$

eliminate $\dot{\varphi}$ from (1) using Conservation of Ho also set $\dot{r}=0$ at that point then $\frac{d}{dt}$ of both sides give

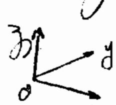
$$\left(2m\ddot{r} - m\frac{r_0^4 \omega_0^2}{r^3} + mg\right)\dot{r} = 0$$

$\ddot{r} = \Rightarrow$ plug into (4) to obtain $\quad R = m\left(\frac{r_0^4 \omega_0^2}{2r^3} + \frac{g}{2}\right)$

R_{max} occures at r_{min} $\quad R_{max} = m\left(\frac{r_0^4 \omega_0^2}{2 r_{min}^3} + \frac{g}{2}\right)$

III Dynamics of Rigid Bodies

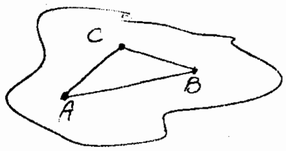
Rigid body



→ Continuum of particles

$|\underline{r}_A - \underline{r}_B| = Const$

for all A & B on the body

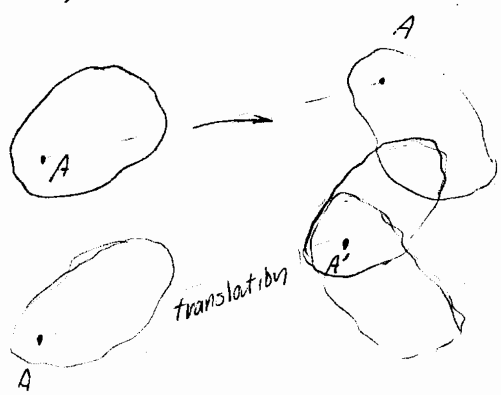


$\# DOF = 3\times3 - \underbrace{3}_{\text{Constraint}} = 6$

$|\underline{r}_A - \underline{r}_B| = Const$
$|\underline{r}_B - \underline{r}_C| = Const$
$|\underline{r}_A - \underline{r}_C| = Const$

General motion of a rigid body

Can always be viewed as a superposition of translation and a rotation about a fixed ~~Axes~~ Point



Session 6

Recall:

- $E = T + V = m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 + mgr = E_0 = \text{Const}$

- $H_0 = m r^2 \dot{\phi} = \text{Const} \Rightarrow \boxed{\dot{\phi} = \frac{H_0}{m r^2}} \quad (*)$

Note: φ is a cyclic Coordinate (ignorable)

$$\frac{\partial E}{\partial \varphi} = 0$$

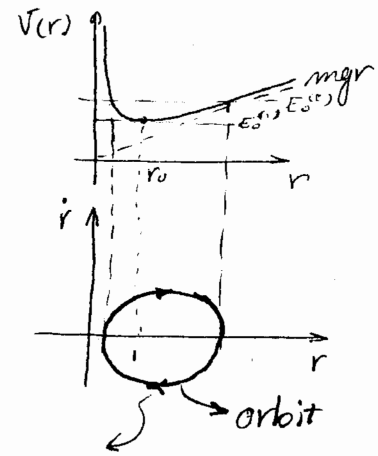
when such a Coordinate present,

DOF Can be reduced by one $\Rightarrow$ reduced mechanical System

In Present Case, use $(*)$ to obtain reduced energy

$$E = \underbrace{m\dot{r}^2}_{T(\dot{r})} + \underbrace{\frac{H_0^2}{2m} \cdot \frac{1}{r^2} + mgr}_{V(r)} = E_0$$

$$\dot{r} = \sqrt{\frac{1}{m}\left(E_0 - V(r)\right)}$$



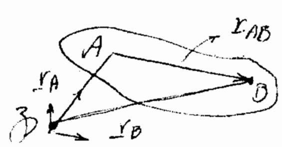
Rigid Body Dynamics

(1) 

$|r_{AB}| = \text{Const}$

(2) # DOF = 6

(3) Velocities at different points of a rigid body



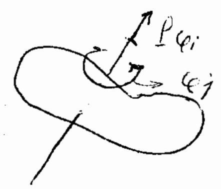
$\underline{r}_B = \underline{r}_A + \underline{r}_{AB}$

$\underline{v}_B = \underline{v}_A + \dot{\underline{r}}_{AB}$

It turns out that there exist a unique vector $\underline{\omega}$ (Angular Velocity of the rigid body) such that

$$\boxed{\underline{v}_B = \underline{v}_A + \underline{\omega} \times \underline{r}_{AB}} \quad \text{for all } A \,\&\, B$$

(2) If the rotation of the rigid body can be instantaneously decomposed to a finite # of rotation about "well-understood" fixed axes, then the angular velocities defined for those rotations, then $\underline{\omega}$ is just the sum of those angular velocities.



$$\Rightarrow \quad \underline{\omega} = \sum_{i=1}^{n} \underline{\omega}_i \quad \Rightarrow$$ the order of summation is unimportant

Surprising, because finite rotation in 3d don't commute

to prove (1), note that instantaneously, B performs an instantaneous rotation about A

In general Rotation in 3D about a fixed point can be described through matrix multiplication.

$$\underline{e}(t) = \underline{\underline{R}}(t)\, \underline{e}_0$$ where $R(t)$ is a $\boxed{\text{proper orthogonal matrix}}$

Main properties of such matrices

(a) Preserve length $|\underline{e}(t)|^2 = |\underline{e}_0|^2$ or $\langle \underline{\underline{R}}(t)\underline{e}_0, \underline{\underline{R}}(t)\,\underline{e}_0 \rangle = \langle \underline{e}_0, \underline{e}_0 \rangle$

In general $\langle \underline{\underline{T}}\,\underline{a}, \underline{b} \rangle = \langle \underline{a}, \underline{\underline{T}}^t \underline{b} \rangle$

transpose of $\underline{\underline{T}}$

$$\langle \underline{e}_0, \underbrace{\underline{\underline{R}}^T(t)\, \underline{\underline{R}}(t)}_{\underline{\underline{I}}} \underline{e}_0 \rangle = \langle \underline{e}_0, \underline{e}_0 \rangle$$

$\underline{\underline{I}}$ (because $\underline{e}_0$ is an identity)

here $\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$

$$\det(R^T)\cdot \det(R) = 1 \quad \rightarrow \quad |\det(R)| = 1$$

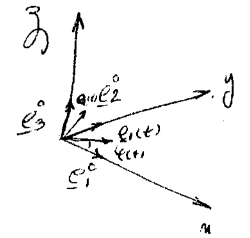
b) Preserve orientation of vectors

$$\Rightarrow \quad \det \underline{\underline{R}} > 0 \quad \Rightarrow \quad \det(R(t)) = 1$$

Example

$$\underline{\underline{R}}(t) = \begin{pmatrix} \cos\varphi(t) & -\sin\varphi(t) & 0 \\ \sin\varphi(t) & \cos\varphi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\underline{e}_1(t)$ $\underline{e}_2(t)$ $\underline{e}_3(t)$



Using the above, fixing A we obtain

$$\frac{d}{dt}\left\{ \underline{r}_{AB}(t) = \underline{\underline{R}}(t)\, \underline{r}_{AB}(0) \right\}$$

$$\dot{\underline{r}}_{AB} = \dot{\underline{\underline{R}}}\, \underline{r}_{AB}(0)$$

Note: $\underline{\underline{R}}\,\underline{\underline{R}}^t = I \quad / \frac{d}{dt}$

$$\dot{\underline{\underline{R}}}\,\underline{\underline{R}}^t + \underline{\underline{R}}\,\dot{\underline{\underline{R}}}^t = 0$$

$$\Rightarrow \dot{\underline{\underline{R}}} = -\underline{\underline{R}}\,\dot{\underline{\underline{R}}}^t\,\underline{\underline{R}}$$

$$\Rightarrow \dot{\underline{r}}_{AB} = \underbrace{-\underline{\underline{R}}\,\dot{\underline{\underline{R}}}^t}_{\underline{\underline{\Omega}}^A(t)}\, \underbrace{\underline{\underline{R}}\,\underline{r}_{AB}(0)}_{\underline{r}_{AB}(t)}$$

$$= \underline{\underline{\Omega}}^A(t)\, \underline{r}_{AB}(t)$$

$\Rightarrow [\underline{\underline{\Omega}}^A(t)]^T = -\underline{\underline{\Omega}}^A(t)$

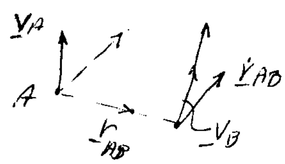
$\Rightarrow \underline{\underline{\Omega}}^A(t)$ skew symmetric

By Assignment #2, for any 3D skew symmetric matrix Ω^A, there exists a 3D vector $\underline{\omega}^A$ such that $\underline{\underline{\Omega}}^A \underline{r} = \underline{\omega}^A \times \underline{r}$

$$\dot{\underline{r}}_{AB}(t) = \underline{\underline{\Omega}}^A(t)\, \underline{r}_{AB}(t)$$
$$= \underline{\omega}_A(t) \times \underline{r}_{AB}(t)$$

but $\dot{\underline{r}}_{AB} = \underline{v}_B - \underline{v}_A$

$$\Rightarrow \boxed{\underline{v}_B = \underline{v}_A + \underline{\omega}_A \times \underline{r}_{AB}}$$



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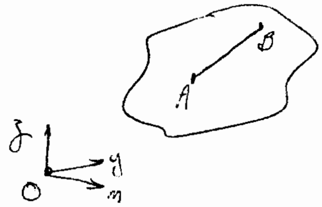
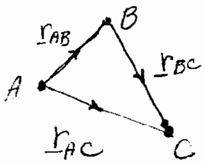


$\underline{v}_B = \underline{v}_A + \underline{\omega}^A \times \underline{r}_{AB}$

To complete the argument started last time we need to show that ω^A is in fact independent of A



$$\underline{v}_C = \underline{v}_A + \underline{\omega}^A \times \underline{r}_{AC}$$
$$\underline{v}_C = \underline{v}_B + \underline{\omega}^B \times \underline{r}_{BC}$$
$$\mapsto \underline{v}_A - \underline{v}_B = \underline{\omega}_B \times \underline{r}_{BC} - \underline{\omega}_A \times \underline{r}_{AC}$$

$$\underline{\omega}^A \times (-\underline{r}_{AB} + \underline{r}_{AC}) = \underline{\omega}^B \times \underline{r}_{BC}$$

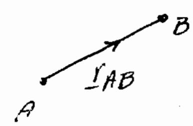
$(\underline{\omega}^A - \underline{\omega}^B) \times \underline{r}_{BC} = 0$ because $\underline{r}_{BC}$ is arbitrary

$\Rightarrow \underline{\underline{\omega}}^A = \underline{\underline{\omega}}^B \overset{\text{def}}{=} \omega$

$\Rightarrow \underline{v}_B = \underline{v}_A + \underline{\omega} \times \underline{r}_{AB}$

(2) Show: ω can be obtained by adding angular velocities about different axes

To see this, fix A instantaneously and consider composition of k rigid body rotation about A



$$\underline{r}_{AB}(t) = \underline{\underline{R}}(t)\, \underline{r}_{AB}(0)$$
$$= \underbrace{(\underline{\underline{R}}_k \ldots \underline{\underline{R}}_2 \underline{\underline{R}}_1)}_{k \text{ rotation}} \underline{r}_{AB}(0)$$

$$\dot{r}_{AB} = (\dot{\underline{\underline{R}}}_k \underline{\underline{R}}_{k-1} \cdots \underline{\underline{R}}_1$$
$$+ \underline{\underline{R}}_k \dot{\underline{\underline{R}}}_{k-1} \cdots \underline{\underline{R}}_1$$
$$+ \underline{\underline{R}}_k \underline{\underline{R}}_{k-1} \cdots \dot{\underline{\underline{R}}}_1)\, \underline{r}_{AB}(0)$$

Recall $\dot{\underline{\underline{R}}}_i = -\underline{\underline{R}}_i \dot{\underline{\underline{R}}}_i^T \underline{\underline{R}}_i$

$$\underline{\dot{r}}_{AB} = (-\underline{\underline{R}}_k \dot{\underline{\underline{R}}}_k^T \underline{\underline{R}}_k \cdots \underline{\underline{R}}_1 + \ldots$$
$$+ \underline{\underline{R}}_k \cdots \underline{\underline{R}}_2 (-\underline{\underline{R}}_1 \dot{\underline{\underline{R}}}_1^T \underline{\underline{R}}_1)$$

Note: $-\underline{\underline{R}}_i \dot{\underline{\underline{R}}}_i^T \big|_{t=0} = \underline{\underline{\Omega}}_i \quad , \quad \underline{\underline{R}}_i(0) = \underline{\underline{I}}$

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$$\Rightarrow \dot{\underline{r}}_{AB}(0) = (\underline{\underline{\Omega}}_K + \underline{\underline{\Omega}}_{K-1} + \cdots + \underline{\underline{\Omega}}_1)\, \underline{r}_{AB}(0)$$

$$= \sum_{i=1}^{K} \underline{\underline{\Omega}}_i\, \underline{r}_{AB}(0) = \sum_{i=1}^{K} \underline{\omega}_i \times \underline{r}_{AB}(0) = \overbrace{\left(\sum_{i=1}^{K} \omega_i\right)}^{\underline{\omega}} \times r_{AB}(0)$$

Conclusion: $\boxed{\underline{\omega} = \sum_{i=1}^{K} \underline{\omega}_i}$

In application, how do we find ω?

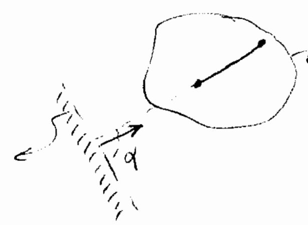
a) Identify all axes about which body rotates then add component angular velocities

b) If we know $\underline{v}_A$ $\underline{v}_C$ (and $\underline{r}_{AC}$) we obtain $\underline{\omega}$ by solving solution it from

$$\underline{v}_C - \underline{v}_A = \underline{\omega} \times \underline{r}_{AC}$$

c) in two-D motion (x-y) plane, $\underline{\omega} = \omega \underline{k}$

then ω can be identified as follows

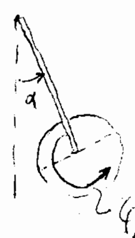


"painted line" on body

Fixed direction in internal frame

then $\omega = \dot{\alpha}$

Eg.

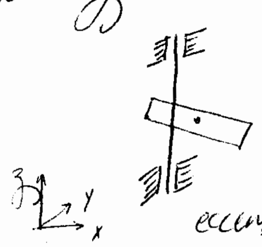


$\underline{\omega} = (\dot{\alpha} + \dot{\psi})\, \underline{k}$

One last word about rigid body rotation

often some vectors are more convenient in frame that rotating with rigid body

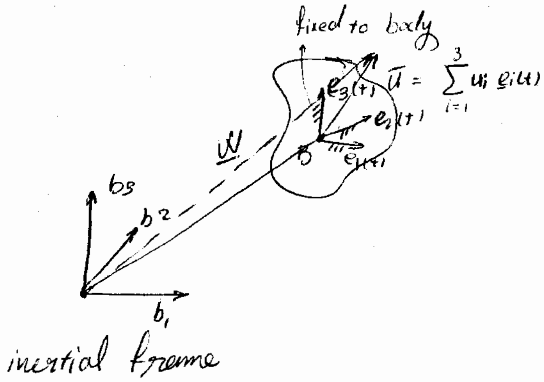
eg.



eccentric and tilted

In this example, as seen later, the angular momentum vector is easier to compute in a rotating frame (ξ, η, ζ)

For cases like this we need to know how to evaluate the "real" (relative to inertial frame) time derivative of vectors in question



$e_i(t)$ is fixed to the body

$$\dot{W} = V_B + \dot{u}$$
$$= V_B + \sum_{i=1}^{3}\left[\dot{u}_i e_i(t) + u_i \dot{e}_i(t)\right]$$
$$= V_B + \underbrace{\sum_{i=1}^{3} \dot{u}_i e_i(t)}_{\overset{\circ}{u}} + \underbrace{\underbrace{\sum_{i=1}^{3} u_i \underbrace{\dot{e}_i(t)}_{\omega \times e_i}}_{\omega \times \sum_{i=1}^{3} u_i e_i}}_{\omega \times u}$$

relative derivative of u in $[e_1, e_2, e_3]$

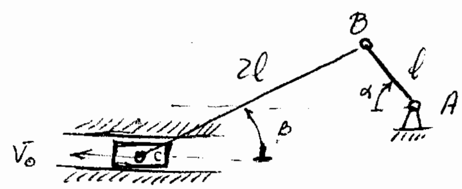
$$\dot{u} = \overset{\circ}{u} + \omega \times u$$

also $$\dot{W} = V_B + \overset{\circ}{u} + \omega \times u$$

Example (2D)

$V_B = ?$

its direction is known to be perpendicular to the AB

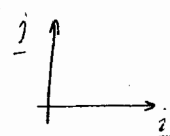


$$\# DOF = 3\times 3 - \underset{\text{joint } A}{2} - \underset{\text{joint } B}{2} - \underset{\text{joint } C}{2} - \underset{\text{Constraint for the block}}{2}$$

joint B: x, y at 2ℓ rod is confined to point B of rod ℓ

V_B can be expanded in terms of say α and $\dot{\alpha}$ (=?) and V_0

$$\left.\begin{aligned} V_B &= \overset{=0}{V_A} + \omega_1 \times r_{AB} \\ V_B &= \underbrace{V_C}_{-V_0 i} + \omega_2 \times r_{CB} \end{aligned}\right\} \quad \omega_1$$



$$\omega_1 = -\dot{\alpha} K$$
$$\omega_2 = \dot{\beta} K \quad \Rightarrow \quad \hat{i}(\ell\dot{\alpha}\sin\alpha + V_0 + 2\ell\dot{\beta}\sin\beta) = \hat{j}(-\dot{\alpha}\ell + 2\ell\dot{\beta}\cos\beta)$$

From geometry $2\ell\sin\beta - a = \ell\sin\alpha \Rightarrow V_B = V_0 \dfrac{\sqrt{4\ell^2 - (a+\ell\sin\alpha)^2}}{\cos\alpha\sqrt{4\ell^2 - (a+\ell\sin\alpha)^2} + a + \ell\sin\alpha}(\cos\alpha\,\hat{j} - \sin\alpha\,\hat{i})$

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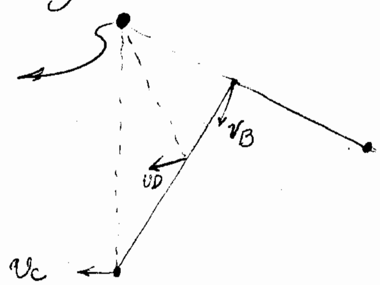
* in 2D if we know v_B and v_A that is enough to determine ω
but in 3D we need at least to know the velocity of 3 points

instantaneous Center
of Rotation

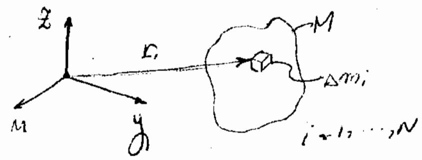


you can get the direction of velocity
by having the instantaneous
Center of Rotation

~~Kinetics~~ in 3 dim, there is an axis of Rotation

Kinetics of Rigid bodies

(1) Linear momentum principle



Define linear momentum as $\underline{P} = \lim_{\substack{N\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{N} \dot{r}_i \, \Delta m_i = \int_M \underline{v} \, dm$

Note: $dm = \rho dV = \rho dx\, dy\, dz$

By taking the limit $N\to\infty$ in our discussion for systems of particles:

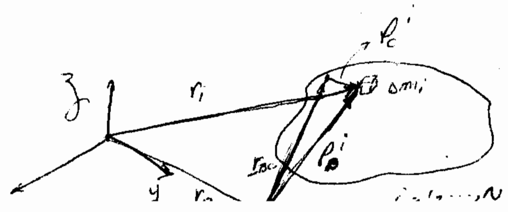
$\boxed{\underline{P} = M \underline{V}_c}$ where c is the Center of mass defined by $r_c = \frac{1}{M}\int_M \underline{r} \, dm$

By def, C is again the point for which the total mass moment vanishes

Also, $N\to\infty$ gives $\boxed{\underline{\dot{P}} = \underline{F}}$ F: resultant external force

. if $F=0 \Rightarrow \underline{P} = \text{Const}$ (Conservation of linear momentum)

(2) Angular momentum principle



Define angular momentum $\quad H_B = \lim_{\substack{N\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{N} \rho_B^i \times (\Delta m_i \dot{r}_i)$

$$= \int_M \rho_B \times \underline{V} \, dm$$

taking the $N\to\infty$ limit in our Calculations in Systems of particles:

$$\dot{H}_B + \underline{V}_B \times \underline{P} = \underline{M}_B$$

where $\underline{M}_B$ is the resultant external torque w.r.t. B

Special Case

if $\underline{M}_B = \underline{0}$ AND $\underline{V}_B = \underline{0}$ or $B = CM$ or $\underline{V}_B \parallel \underline{V}_C$

then $\quad \underline{H}_B = \text{Const} \quad$ Conservation of angular momentum

How do we Compute $\underline{H}_B$?

First Note that if c is the CM, then $H_B = \int_M (\underline{r}_{BC} + \rho_c) \times (\underline{V}_c + \underline{\omega} \times \rho_c)\, dm$

$$= \underline{r}_{BC} \times (\underline{V}_c M) + \underline{r}_{BC} \times \underline{\omega} \times \int_M \rho_c \, dm$$

$$+ \underbrace{\int_M \rho_c \times (\underline{V}_c + \underline{\omega} \times \rho_c)\, dm}_{H_C}$$

$$\Rightarrow \boxed{\underline{H}_B = \underline{H}_C + \underline{P} \times \underline{r}_{CB}} \quad (*)$$

Similar to $\quad V_B = V_A + \omega \times r_{AB}$

where $\boxed{\underline{H}_C = \int_M \rho_c \times \underline{\omega} \times \rho_c \, dm} \rightarrow$ Centroidal angular momentum

Note: for any other point A

$$(**) \quad \underline{H}_A = \underline{H}_C + \underline{P} \times \underline{r}_{CA}$$

then $(*)$ and $(**)$ $\Rightarrow$

$$\boxed{\underline{H}_B = \underline{H}_A + \underline{P} \times \underline{r}_{AB}}$$

Computation of $\underline{H}_C$

Note: $\quad \rho_c \times (\underline{\omega} \times \rho_c) = (\rho_c \cdot \rho_c)\, \underline{\omega} - (\rho_c \cdot \underline{\omega})\, \rho_c$

$$(\underline{a} \times \underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}$$

$\Rightarrow \rho_c \times (\omega \times \rho_c) = $ ((x,y,z) to CM)

$$\begin{pmatrix} x^2+y^2+z^2 & 0 & 0 \\ 0 & x^2+y^2+z^2 & 0 \\ 0 & 0 & x^2+y^2+z^2 \end{pmatrix} \underline{\omega} - \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} \underline{\omega}$$

$$= \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \underline{\omega}$$

$\Rightarrow \underline{H}_c = \underline{\underline{I}}_c \, \underline{\omega}$

where $\underline{\underline{I}}_c$ is the <u>Centroidal moment of inertia tensor</u> defined as:

$$\underline{\underline{I}}_c = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix}$$

with $I_{xx} = \int_M (y^2+z^2)\,dm$

$I_{xy} = -\int_M xy\,dm \qquad I_{yy} = \int_M (x^2+z^2)\,dm$

$I_{xz} = -\int_M xz\,dm \qquad I_{zz} = \int_M (x^2+y^2)\,dm$

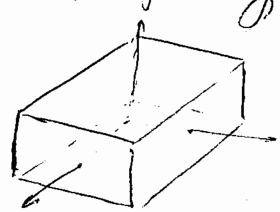
$I_{yz} = -\int_M yz\,dm$

properties of $\underline{\underline{I}}_c$

- Symmetric $\Rightarrow$ 3 real
- eigenvalues orthogonal
- eigen vectors (principal axes of inertia)

In the principal axes frame $\underline{\underline{I}}_c = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

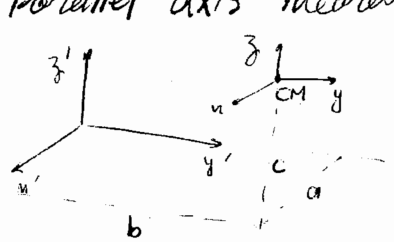
Note: axes of Symmetry are Automatically principal axes



- $\underline{\underline{I}}_c$ positive definite, i.e. $I_1, I_2, I_3 > 0$ (will See later why)
- In 2D $\quad \underline{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \omega \underline{K}$

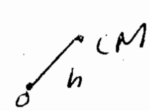
$$\Rightarrow \boxed{\underline{H}_c = I_{zz}\, \omega \, \underline{K}}$$

- Parallel axis theorem (xyz attached to CM)



then $\underline{\underline{I}}_o = \underline{\underline{I}}_c + M \begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}$

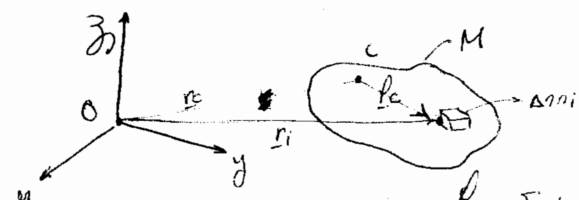
in 2D. $I_o = I_c + Mh^2$



$\underline{H}_c = \underline{\underline{I}}_c\, \underline{\omega}$
↳ moment of inertia tensor

$\underline{H}_B = \underline{\underline{I}}_B\, \underline{\omega}$

(3) Work-Energy Principle



$$T = \sum_{i=1}^{N} \frac{1}{2} \Delta m_i |\dot{r}_i|^2 \underset{N\to\infty}{=} \frac{1}{2}\int_M |\underline{v}|^2 dm$$

also using the limit of the argument given for systems of particles, $W_{12} = T_2 - T_1$

work done by external forces (rigid body)

If all forces are potential, $T+V = \text{Const}$ Conservation of energy

Conservation of Energy

To evaluate T let $\underline{v} = \underline{v}_c + \underline{\omega} \times \underline{\rho}_c$

$$\Rightarrow T = \frac{1}{2}\int_M [\underline{v}_c + \underline{\omega}\times\underline{\rho}_c]^2\, dm$$

$$= \frac{1}{2}\int_M [2\underline{v}_c \cdot (\underline{\omega}\times\underline{\rho}_c) + |\underline{v}_c|^2 + (\underline{\omega}\times\underline{\rho}_c)\cdot(\underline{\omega}\times\underline{\rho}_c)]\, dm$$

$$= \frac{1}{2} M |V_c|^2 + \underline{v}_c \cdot (\underline{\omega} \times (\overset{0}{\rho_c dm})) + \frac{1}{2}\int_M (\underline{\omega}\times\underline{\rho}_c)\cdot(\underline{\omega}\times\underline{\rho}_c)\, dm$$

$(\underline{a}\times\underline{b})\cdot\underline{c} = (\underline{a}\ \underline{b}\ \underline{c})$

Note: $\underset{a}{\underline{\omega}} \times \underset{b}{\underline{\rho}_c} \cdot \underset{c}{(\underline{\omega}\times\underline{\rho}_c)} =$ $(\underline{b}\times\underline{c})\cdot\underline{a}$ $\underline{\rho}_c \times (\underline{\omega}\times\underline{\rho}_c)\cdot\underline{\omega}$

$$T = \frac{1}{2} M |\underline{V}_c|^2 + \frac{1}{2}(\underline{\underline{I}}_c\,\underline{\omega})\cdot\underline{\omega}$$

$$\boxed{T = \frac{1}{2} M |\underline{V}_c|^2 + \frac{1}{2}\underline{\omega}^T \underline{\underline{I}}_c\,\underline{\omega}}$$

$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$

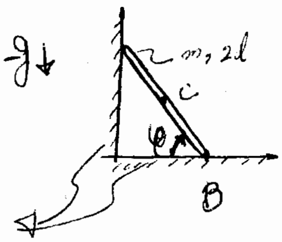
translational part — Rotational part

Assume that $\underline{V}_c = 0$ (CM is fixed) then $T > 0$ must hold for any $\underline{\omega} \Rightarrow \underline{\underline{I}}_c$ must be positive definite.

Assume not $\Rightarrow$ principal basis $T = \frac{1}{2}(\omega_1^2 I_1 + \omega_2^2 I_2 + \omega_3^3 I_3)$

Final Note: if $v_B = 0 \Rightarrow \boxed{T = \frac{1}{2}\underline{\omega}^T \underline{\underline{I}}_B \underline{\omega}}$

Example 1:



Smooth

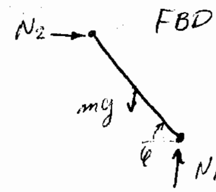
- Equation of Motion?
- Reaction forces?

$\# DOF = 3 - 2 \times 1 = 1$

unconstraint #DOF; # of Constraints

$\Rightarrow$ use 1 generalized coordinate (φ)

Linear momentum principle



$$\dot{\underline{P}} = \underline{F} = N_2 \underline{i} + (N_1 - mg)\underline{j}$$

$$\dot{\underline{P}} = \frac{d}{dt}(m \underline{v}_c) = m \underline{a}_c$$

To compute a_c: $\underline{v}_c = \underline{v}_B + \underline{\omega} \times \underline{r}_{BC}$

$$= \frac{d}{dt}(2l\cos\varphi)\,\underline{i} + (-\dot\varphi \underline{k}) \times (-l\cos\varphi\, \underline{i} + l\sin\varphi\, \underline{j})$$

$$= l\dot\varphi(\sin\varphi\, \underline{i} + \cos\varphi\, \underline{j})$$

$$\Rightarrow \underline{a}_c = (l\dot\varphi^2\cos\varphi + l\ddot\varphi\sin\varphi)\,\underline{i} + (-l\dot\varphi^2\sin\varphi + l\ddot\varphi\cos\varphi)\,\underline{j}$$

(1) $m l\ddot\varphi\sin\varphi + m l\dot\varphi^2\cos\varphi = N_2$

(2) $m l\ddot\varphi\cos\varphi - m l\dot\varphi^2\sin\varphi = N_1 - mg$

Angular momentum principle w.r.t. B

$$\dot{\underline{H}}_B + \underline{v}_B \times \underline{P} = \underline{M}_B = (mgl\cos\varphi - N_2\, 2l\sin\varphi)\,\underline{k}$$

$$\underline{H}_B = \underline{H}_C + \underline{P} \times \underline{r}_{CB}$$

$$= \underline{I}_C\,\underline{\omega} + \underline{P} \times \underline{r}_{CB}$$

$$= \frac{1}{12} m (2l)^2 (-\dot\varphi)\,\underline{k} - m(l\dot\varphi\sin\varphi\, \underline{i} + l\dot\varphi\cos\varphi\, \underline{j}) \times (l\cos\varphi\, \underline{i} - l\sin\varphi\, \underline{j})$$

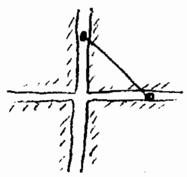
$$\Rightarrow H_B = \frac{4}{3} m l^2 \dot\varphi\, \underline{k}$$

(3) $\Rightarrow \dot{\underline{H}}_B + \underline{v}_B \times \underline{P} = \left(\frac{4}{3} m l^2 \ddot\varphi - 2ml^2\dot\varphi^2\sin\varphi\cos\varphi\right)\underline{k}$

(3): $\frac{4}{3} m l^2 \ddot{\varphi} - 2 m l^2 \dot{\varphi}^2 \sin\varphi \cos\varphi = m g l \cos\varphi - N_2 2 l \sin\varphi$

(1), (3) $\longrightarrow$ $\boxed{m l^2 \left(\frac{4}{3} + 2 \sin^2\varphi\right) \ddot{\varphi} - m g l \cos\varphi = 0}$

Eq of motion



Since energy is Conserved (active force in potential, Constraints forces do not work)

$T + V = \text{Const}$

$\frac{d}{dt}(T+V) = 0$ Solve eq of motion

Reaction forces

$$N_2 = \frac{3 m g \sin 2\varphi}{4(2 + 3\sin^2\varphi)} + m l \dot{\varphi} \cos\varphi$$

Note: $\dot{\varphi}$ Can be expressed as a function of φ from $T + V = \underbrace{T_0 + V_0}_{\text{total Energy}}$

$$N_1 = \frac{3 m g \cos^2\varphi}{2(2 + 3\sin^2\varphi)}$$

$$- m l \dot{\varphi}^2 \sin\varphi + m g$$

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Session 10

$\Rightarrow L = \frac{1}{6}ML^2\dot{\varphi}^2 + \frac{1}{2}m[\dot{r}^2 + r^2\dot{\varphi}^2] + g\left(\frac{ML}{2} + mr\right)\cos\varphi$

= Recall from Virtual work $\Rightarrow Q_\varphi^{\text{non-potential}} = Fl\sin\alpha$

$Q_r^{\text{non-pot}} = \mu m\,|r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + g\sin\varphi|\cdot\text{Sign}(\dot{r})$

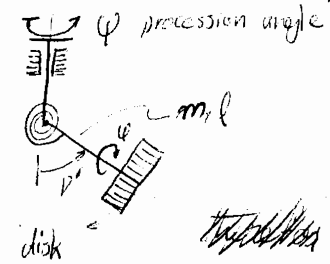
$\rightarrow$ Equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} = Q_\varphi \Rightarrow \left(\frac{1}{3}ML^2 + mr^2\right)\ddot{\varphi} + 2mr\dot{r}\dot{\varphi} + g\left(\frac{ML}{2} + mr\right)\sin\varphi = Fl\sin\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{r}} - \frac{\partial L}{\partial r} = Q_r \Rightarrow m\ddot{r} - mr\dot{\varphi}^2 - mg\cos\varphi = -\mu m|r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + g\sin\varphi|\cdot\text{Sign}(\dot{r})$$

EXAMPLE illustrates frame independence of Lagrangian approach

$g\downarrow$ Torsional Spring (k)



Assume: Spring is unstretched for $\nu = 0$

DOF: $2\times6 - 5 - 4 = 3$

~~[illegible]~~ System is holonomic

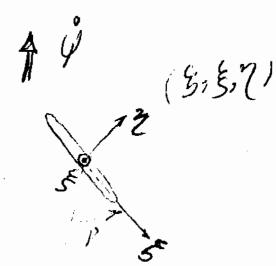
Lagrangian equation of motion applies

apply $L(\psi, \nu, \phi, \dot{\psi}, \dot{\nu}, \dot{\phi})$

- Forces – Constraint forces are ideal
- Active forces are potential $\longrightarrow \underline{Q}_q = \underline{0}$

$T = T^{beam} + T^{disk}$; $V = V^{beam} + V^{disk} + V^{spring}$

$T^{beam} = \frac{1}{2}m|V_c|^2 + \frac{1}{2}\underline{\omega}^{beam}\,\underline{\underline{I}}_c\,\underline{\omega}^{beam}$



Consider a frame which is principal fix for beam.

$\Rightarrow T_{beam} = \frac{1}{2}m\left[\left(\frac{L}{2}\sin\nu\,\dot{\psi}\right)^2 + \left(\frac{L}{2}\dot{\nu}\right)^2\right]$

$+ \frac{1}{2}\left[I^b_{\xi\xi}(-\dot{\psi}\cos\nu)^2 + I^b_{\zeta\zeta}(-\dot{\nu})^2 + I^b_{\eta\eta}(\dot{\psi}\sin\nu)^2\right]$

$I^b_{\xi\xi} = 0$ (Slender beam) $I^b_{\eta\eta} = I^b_{\zeta\zeta} = \frac{1}{12}mL^2$

$T^{beam} = \frac{1}{6}mL^2(\sin^2\nu\,\dot{\psi}^2 + \dot{\nu}^2)$

$T^{disk} = \frac{1}{2}M|V_D|^2 + \frac{1}{2}(\underline{\omega}^{disk})^T\,\underline{I}_c\,\underline{\omega}^{disk}$

Use previous frame shifted to B

$$\Rightarrow T_{disk} = \frac{1}{2} M\left[(L \sin\nu \dot{\psi})^2 + (L\dot{\nu})^2\right]$$
$$+ \frac{1}{2}\left[I^d_{\xi\xi}(-\dot{\varphi} - \dot{\psi}\cos\nu) + I^d_{\xi\xi}(-\dot{\nu})^2 + I^d_{\eta\eta}(\dot{\psi}\sin\nu)^2\right]$$

Note: $I^d_{\xi\xi} = \frac{1}{2}MR^2$, $I^d_{\xi\xi} = I^d_{\eta\eta} = \frac{1}{4}MR^2$

$$\Rightarrow T^{disk} = \frac{1}{2}M\left[L^2 + \frac{1}{4}R^2\right]\sin^2\nu \dot{\psi}^2 + \frac{1}{2}M(L^2 + R^2)\dot{\nu}^2 + \frac{1}{4}MR^2(\dot{\varphi} + \dot{\psi}\cos\nu)^2$$

$$V = -mg\frac{L}{2}\cos\nu - MgL\cos\nu + \frac{1}{2}K\nu^2$$

$$L = T - V = \ldots$$

$\Rightarrow$ Eq of motion $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$

$q = \psi, \nu, \varphi$

φ, ψ are called cyclic Coordinate (ignorable)

Symmetry

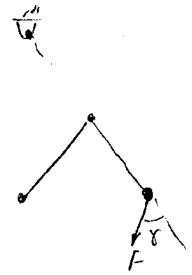
Lagrangian doesn't depend on them explicitly

Conservation of angular momentum $\leftarrow$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = 0$, $\quad \frac{\partial L}{\partial \psi} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\nu}}\right) - \frac{\partial L}{\partial \nu} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$$

[redacted]
Session 11



F: follower force

$$F = -\nabla V \Leftarrow \begin{cases} F_x = -\frac{\partial V}{\partial x} \\ F_y = -\frac{\partial V}{\partial y} \end{cases}$$

$$\frac{\partial}{\partial y} F_x = \frac{\partial}{\partial x} F_y$$

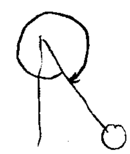
$\mapsto$ F is not potential

works done by F

$$W_{12} = |F| \sin\gamma \, L(\varphi_2 - \varphi_1)$$

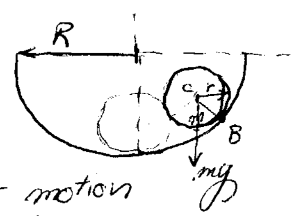
Note W_{12} is not path-independent

This system is locally Conservative



Example

$g \downarrow$



1 DOF = 3-2=1 $\rightarrow$ choose φ as a generalized Coordinate

Question. Eq of motion
Reaction forces

To eliminate the role of Constraint forces, take ang momentum principle w.r.t. B

$$\dot{H}_B + V_B \times P = M_B$$

$$M_B = mgr \sin\varphi \, k$$

$$V_B \parallel V_C \Rightarrow V_B \parallel P$$

$$H_B \neq I_B \omega \, !$$

$$\Rightarrow H_B = H_C + P \times r_{CB}$$

$$= I_C \omega k + P \times r_{CB}$$

$$= \frac{1}{2} m r^2 (-\dot{\theta}) k + m(R-r)\dot{\varphi} (\cos\varphi \, \hat{i} + \sin\varphi \, \hat{j}) \times r (\sin\varphi \, \hat{i} - \cos\varphi \, \hat{j})$$

Note: $R\varphi = r(\theta + \varphi) \Rightarrow \dot{\theta} = \frac{R-r}{r} \dot{\varphi}$

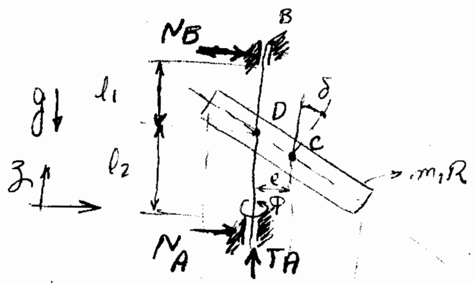
$$\Rightarrow H_B = -\frac{3}{2} m r (R-r) \dot{\varphi} \, k$$

$\Rightarrow$ Eq. of motion $-\frac{3}{2} m r (R-r) \ddot{\varphi} = mgr \sin\varphi \Rightarrow \ddot{\varphi} + \frac{2g}{3(R-r)} \sin\varphi = 0$

For frequency of Small oscillations linearized $\rightarrow \ddot{\varphi} + \frac{2g}{3(R-r)} \varphi = 0 \qquad \alpha = \sqrt{\frac{2g}{3(R-r)}}$

Example

eccentric skewed disk on a rotating shaft

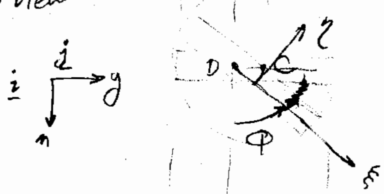


Reaction forces at A & B?

$\# DOF = 6-3-2=1 \rightarrow$ use φ

Linear momentum principle:

top view



$$\dot{P} = ma = (N_A+N_B)\cos\varphi\,\hat{i} + (N_A+N_B)\sin\varphi\,\hat{j} + (T_A - Mg)\hat{k}$$

$$r_C = e(\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}) + z_C\,\hat{k}$$

$$\Rightarrow a_C = -e(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi)\hat{i} + e(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi)\hat{j}$$

$\Rightarrow$ linear momentum principle:

$$-em(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) = (N_A+N_B)\cos\varphi$$
$$em(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) = (N_A+N_B)\sin\varphi$$
$$0 = T_A - mg \longrightarrow \boxed{T_A = Mg}$$

$$\Rightarrow -em\dot{\varphi}^2 = N_A + N_B$$

Angular momentum principle: $\dot{\underline{H}}_C + \underline{v}_C \times \underline{P} = \underline{M}_C$

$$\underline{M}_C = N_B(l_1 + e\tan\delta) - N_A(l_2 - e\tan\delta) + T_A e$$

$$\dot{\underline{H}}_C = \overset{\circ}{\underline{H}}_C + \underline{\omega} \times \underline{H}_C$$

relative to frame

$$\underline{H}_C = \underline{\underline{I}}_C\,\underline{\omega} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}\begin{pmatrix} -\dot{\varphi}\sin\delta \\ 0 \\ \dot{\varphi}\cos\delta \end{pmatrix}$$

$$I_1 = I_2 = \frac{1}{4}m\left[R^2 + \frac{h^2}{3}\right].$$
$$I_3 = \frac{1}{2}mR^2$$

$$= \begin{Bmatrix} -I_1\dot{\varphi}\sin\delta \\ 0 \\ I_3\dot{\varphi}\cos\delta \end{Bmatrix} \Rightarrow \dot{\underline{H}}_C = \overset{\circ}{\underline{H}}_C + \omega \times H_C = \begin{Bmatrix} -I_1\ddot{\varphi}\sin\delta \\ 0 \\ I_3\ddot{\varphi}\cos\delta \end{Bmatrix} + \begin{vmatrix} e_1 & e_2 & e_3 \\ -\dot{\varphi}\sin\delta & 0 & \dot{\varphi}\cos\delta \\ -I_1\dot{\varphi}\sin\delta & 0 & I_3\dot{\varphi}\cos\delta \end{vmatrix}$$

$$= \begin{pmatrix} -I_1\ddot{\varphi}\sin\delta \\ -\dot{\varphi}^2\sin\delta\cos\delta(I_1 - I_3) \\ I_3\ddot{\varphi}\cos\delta \end{pmatrix}$$

Session 12

Finish the example

In $[e_1\ e_2\ e_3]$

$$-I_1 \ddot{\varphi} \sin\delta = 0$$

$$\frac{1}{2}\dot{\varphi}^2 (I_3 - I_1) \,\mathcal{S}\delta = N_B(\ell_1 + e\tan\delta) - N_A(\ell_2 - e\tan\delta) + T_A e$$

$$I_3 \ddot{\varphi} \cos\delta = 0$$

$$\Rightarrow \ddot{\varphi} = 0 \Rightarrow \dot{\varphi} = \omega_2 - \text{Const.} \quad \Rightarrow \varphi(t) = \varphi_0 + \omega_0 (t - t_0)$$

$$\Rightarrow \frac{1}{2}(I_3 - I_1)\omega_0^2 \,\mathcal{S}\delta = N_B(\ell_1 + e\tan\delta) - N_A(\ell_2 - e\tan\delta) + T_A e$$

Linear momentum Principle: $\begin{cases} -em\,\omega_0^2 = N_A + N_B \\ T_A = mg \end{cases}$

$$\Rightarrow N_B = \frac{\frac{1}{8}m\left(R^2 - \frac{h^2}{3}\right)\omega_0^2 \sin 2\delta - em\left[g + \omega_0^2(\ell_2 - e\tan\delta)\right]}{\ell_1 + \ell_2}$$

$$N_A = \frac{em\left[g - \omega_0^2(\ell_1 + e\tan\delta)\right] - \frac{1}{8}m\left(R^2 - \frac{h^2}{3}\right)\omega_0^2 \,\mathcal{S}\, 2\delta}{\ell_1 + \ell_2}$$

Final Example on Newtonian Mechanics : Gyroscopes

usual requirements in the definition of a gyroscope

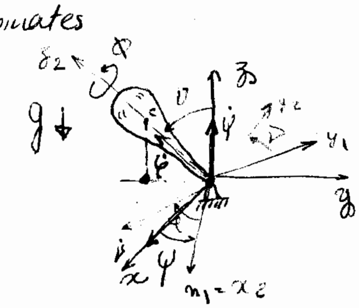
(a) 3D rigid body with one of its point fixed

(b) In principal Coordinates, rotational symmetry is often assumed

$$\underline{\underline{I}}_c \cong \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}$$

(c) angular momentum about 3rd principal axes ($C\omega_3$) dominates

Euler axes & Euler angles use "3-1-3" Convention



"3" : rotation about 3rd axis (z) by angle ψ (precession)

"1" : rotation about 1st axis (x_1) by angle ν (nutation)

"3" : rotation about 3rd axis (z_2) by angle φ (spin)

$$\omega = \dot{\underline{\psi}} + \dot{\underline{\nu}} + \dot{\underline{\varphi}}$$

ANGULAR MOMENTUM PRINCIPLE (about c)

$$\dot{H}_c + \underbrace{\cancel{v_c \times P}}_{0} = M_c$$

→ torque of reaction force at O

Express H_c in principal coordinate

$$\vec{H}_c = \underline{\underline{I}}_c\, \underline{\omega}\ ; \quad \cancel{\vec{\omega} = \begin{bmatrix} \dot{\nu} \\ \dot{\psi}\sin\nu \\ \dot{\varphi}+\dot{\psi}\cos\nu \end{bmatrix}} \qquad \vec{\omega} = \begin{Bmatrix} \dot{\nu}\cos\phi + \dot{\psi}\sin\nu\sin\phi \\ \dot{\psi}\sin\nu\cos\phi - \dot{\nu}\sin\phi \\ \dot{\varphi} + \dot{\psi}\cos\nu \end{Bmatrix}$$

← Correct answer

$$= \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

$$\dot{\underline{H}}_c = \overset{\circ}{H}_c + \vec{\omega} \times \vec{H}_c$$

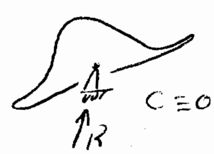
$$= \begin{Bmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{Bmatrix} + \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix} \Rightarrow \boxed{\begin{cases} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = M_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = M_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = M_3 \end{cases}}$$

M_i: expressed in principal coordinate

Euler eq. for spinning top (euler's top)

Special Case $M_i = 0 \quad i = 1, 2, 3$

$$\begin{cases} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0 \\ I_2\dot{\omega}_2 + (I_1 - I_2)\omega_1\omega_3 = 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0 \end{cases}$$



NOTE $\dot{\underline{H}}_c = \overset{\circ}{\underline{H}}_c = \text{Const.}$

(2) $\dot{E} = \dot{T} + \overset{0}{\cancel{\dot{V}}} = 0$

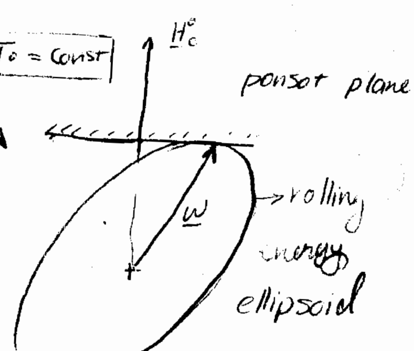
Reaction Force does no work $\quad E = E_0 = T_0 = \text{Const}$

$$T = \frac{1}{2}m|v_c|^2 + \frac{1}{2}\underline{\omega}^T \overbrace{\underline{\underline{I}}_c\,\underline{\omega}}^{H_c} = E_0 = T_0 \Rightarrow \boxed{\underline{H}_c \cdot \underline{\omega} = 2T_0 = \text{Const}}$$

$$\Rightarrow I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T_0$$

$$\frac{\omega_1^2}{\left(\frac{2T_0}{I_1}\right)} + \frac{\omega_2^2}{\left(\frac{2T_0}{I_2}\right)} + \frac{\omega_3^2}{\left(\frac{2T_0}{I_3}\right)} = 1 \longrightarrow \text{Energy Ellipsoid}$$

$\underline{\omega}$ must "run"

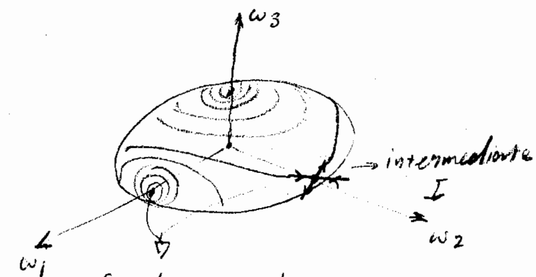


$\Rightarrow$ trajectories (orbits) of euler's spinning top form curves on the energy ellipsoid

trajectories on ellipsoid are called "polliods"

$I_1 < I_2 < I_3$

$\omega_1 = 0$	$\omega_1 = $ Conste	$\omega_1 = 0$
$\omega_2 = $ Cte	$\omega_2 = 0$	$\omega_2 = 0$
$\omega_3 = 0$	$\omega_3 = 0$	$\omega_3 = $ Const



fixed points of the eq of motion

Rotation about intermediate axis is unstable
others are stable

Fixed points (equilibria) for moment-free top

(1) $\omega_1 = \omega_2 = 0, \ \omega_3 \neq 0$ $(\pm)$ (from energy conservation we got two answers

(2) $\omega_1 = \omega_3 = 0 \quad \omega_2 \neq 0$ $(\pm)$

(3) $\omega_2 = \omega_3 = 0 \quad \omega_1 \neq 0$ $(\pm)$

Linearized eq. of motion

$$\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cancel{\frac{I_2 - I_3}{I_1}} & \frac{I_2 - I_3}{I_1}\omega_{30} & \frac{I_2 - I_3}{I_1}\omega_{20} \\ \frac{I_3 - I_1}{I_2}\omega_{30} & 0 & \frac{I_3 - I_1}{I_2}\omega_{10} \\ \frac{I_1 - I_2}{I_3}\omega_{20} & \frac{I_1 - I_2}{I_3}\omega_{10} & 0 \end{bmatrix}}_{\text{jacobian}} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

nonlinear terms
at equilibria

$$(1) \Rightarrow J = \begin{bmatrix} 0 & \frac{I_2 - I_3}{I_1}\omega_{30} & 0 \\ \frac{I_3 - I_1}{I_2}\omega_{30} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eigen values: $\lambda_1 = 0$

$$\lambda_2, \lambda_3 = \pm \sqrt{\frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}}\,\omega_{30} = \pm i\alpha$$

Oscillations about ω_3 axis.

$(2) \Rightarrow \lambda_1 = 0$

$$\lambda_{2,3} = \pm \sqrt{\frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_2}} = \pm \beta$$

Saddle type behavior about ω_2 axis

(3) Similar

Session 13

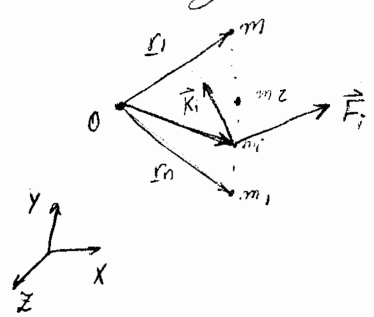
Variational Aproach to dynamics

(Lagrangian mechanics, energy Principle)

- Scalar (work-energy-based) (as opposed to vector based)
- Frame invariant
- renders equations of motion without (in most cases) getting the Constraind reaction forces involved
- Reaction forces are not immediately available (disadvantage)

INGRADIANTS

i) Generalized Coordinates



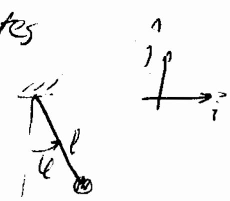
$\vec{F}_i$. active force
$\vec{K}_i$: Constraint force

Often it is possible to select a smaller set of Coordinate (not necessarily position) that uniquely determine the position of the system and already account for (some of the) Construction Paints

Eg. q_i : generalized Coordinates

$x_i = x_i(q_1, \dots, q_N)$
$y_i = y_i(- \quad -)$
$z_i = z_i(- \quad -)$

$y = l\cos\varphi$
$x = l\sin\varphi$



NOTE q_i is a Complete set of Coordinates, but not necessarily independent

ii) Constraints Scalar relations that limit possible motions of the System

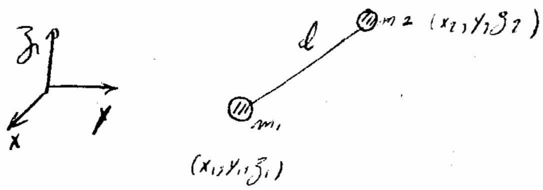
$f_j(r_i, \dot{r}_i, t) = 0 \qquad j = 1, \dots, m$

types of Constraints

types of Constraint	scleronomic	rheonomic (time dependent)
holonomic	$f(r_i) = 0$	$f(r_i, t) = 0$
non-holonomic	$f(r_i, \dot{r}_i) = 0$	$f(r_i, \dot{r}_i, t)$

Example dumbrell

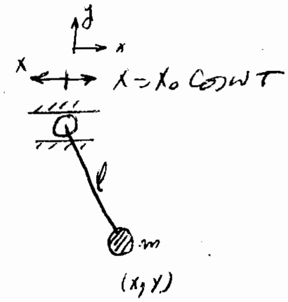
(1) Moving ~~dumbrell~~



$f = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - \ell^2 = 0$

holonomic scleronomic Constraint

(2) Pendulum with Oscillating Support



$(X - x_0 \cos \omega t)^2 + y^2 - \ell^2 = 0$

holonomic Rheonomic Constraint

II Analytical Mechanics

Ingredients (1) generalized Coordinates $r_i = r_i(q_1, \dots, q_n, t)$ $i = 1, \dots, n$

Complete

(2) Constraints $f_j(q, \dot{q}, t) = 0$ (not necessarily independent)

$j = 1, \dots, m$

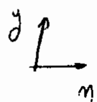
holonomic scleronomic

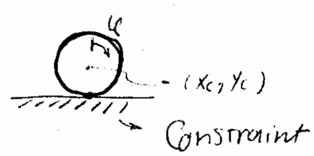
non-holonomic rheonomic

$q = \begin{Bmatrix} q_1 \\ q_n \end{Bmatrix}$

$\# DOF = $ unConstraint $\# DOF - m$
$(3n)$

Example 3, 2D rolling





$\# DOF = 3 - 2 = 1$

$v_p = 0 \Rightarrow v_p = v_c + \omega \times r_{cp}$

$\Rightarrow \left.\begin{array}{l} \dot{x} - R\dot{\varphi} = 0 \\ \dot{y} = 0 \end{array}\right\}$ $(m = 2)$

Strictly Speaking (*) gives 2 nonholonomic Scleronomic Constraints

But $\int dt$ gives

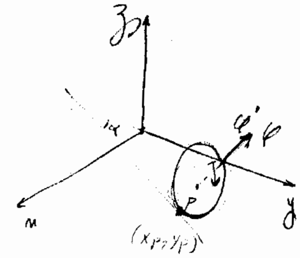
$\left.\begin{array}{l} (x_c - R\varphi) - (x_c^0 - (R\varphi^0)) = 0 \\ y_c = y_{c0} = R \end{array}\right\}$ integrated nonholonomic Constraint (Semi holonomic)

$\begin{cases} x_c = x_c(\varphi) \\ y_c = y_c(\varphi) = R \end{cases}$

a Complete and independent Set of generalized Coardinate is $\{\varphi\}$

Example 4

Vertical disk rotating on 2D plane



$\# DOF = 6 - 2 - 2 = 2$

already accounted by the choice of $(x_p, y_p, \alpha, \varphi)$ ← $\cdot\ z_p = 0$; $\cdot$ rotation angle $= 0$; αP: $v_x = \ldots$, $v_y = 0$

writing out the $v_P^{(x,y)} =$

$$0 = v_P^{(x,y)} = \underbrace{\dot{x}_P \hat{i} + \dot{y}_P \hat{j}}_{v_C} + \underbrace{(-\dot{\varphi}\cos\alpha\,\hat{i} + \dot{\varphi}\sin\alpha\,\hat{j} + \dot{\alpha}K)}_{\omega} \times (-R\,\underline{K})$$

$$\Rightarrow \begin{cases} \dot{x}_P - \dot{\varphi} R \sin\alpha = 0 \\ \dot{y}_P - \dot{\varphi} R \cos\alpha = 0 \end{cases}$$ non holonomic-Sclenomic

INTEGRABILITY OF CONSTRAINTS (digression)

Linear non holonomic Constraints

$$\sum_{i=1}^{N} a_{ji}(q,t)\dot{q}_i + b_j(q,t) = 0 \quad j = 1, \ldots, m \quad (*)$$

Note that most Constraints in mechanics are linear in this Sense Exception e.g.

Note any holonomic Constraint $\{f_j(q,t) = d_j\}_{j=1,\ldots m}$ Can also be write in this form.

$$f_j(q_i(t), t) = 0 \Rightarrow \sum_{i=1}^{N} \frac{\partial f_j}{\partial q_i}\dot{q}_i + \frac{\partial f_j}{\partial t} = 0 \quad j = 1, \ldots m \quad (**)$$

Question: For $(*)$, is there any $\{f_j\}_{j=1}^{m}$ Such that $(*)$ Can be written in the form of $(**)$

Assume. Constraint is

$$\left.\begin{aligned} a_{ji} &= \frac{\partial f_j}{\partial q_i} \\ b_j &= \frac{\partial f_j}{\partial t} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \frac{\partial a_{ji}}{\partial q_k} &= \frac{\partial a_{jk}}{\partial q_i} \\ \frac{\partial a_{ji}}{\partial t} &= \frac{\partial b_j}{\partial q_i} \end{aligned}}$$ For any i, j, k.

Remaining issue $(*)$ may become integrable after multiplication by integrating factor $c_j(q,t)$

$$\Rightarrow \sum \underbrace{(a_j\, c_j)}_{\tilde{a}_{ij}}\dot{q}_i + \underbrace{c_j b_j}_{\tilde{b}_j} = 0$$

$\Rightarrow \underline{v}_P^{(xy)} = \underline{0}$ Constraint is truly non-holonomic and we can not, in obvious way, reduce the # of generalized coordinate

$\Rightarrow$ need to use L_1 coordinates in eq. of motion.

(3) Virtual Displacement
= infinitesimal displacements instantaneously compatible with the constraints

Notation: $\delta \underline{r}_i ; \delta q_k$
Eq for a holonomic set of constraints

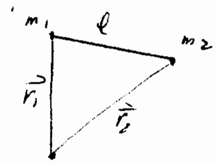
$$f_j(q_1, \ldots, q_n, t) = 0 \quad j = 1, \ldots, m$$

$$\sum_{l=1}^{N} \frac{\partial f_j}{\partial q_l} \delta q_l = 0 \quad j = 1, \ldots, m$$

time is not varied

Example Dumbbell

$$(\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2) - \ell^2 = 0 \qquad \sum_{l=1}^{N} \frac{\partial f_j}{\partial q_l} \delta q_l = 0$$



For the virtual displacement

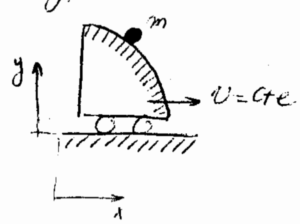
$$2(\underline{r}_1 - \underline{r}_2) \cdot (\delta \underline{r}_1 - \delta \underline{r}_2) = 0$$

$\Rightarrow \delta r_1 - \delta r_2 \perp (r_1 - r_2)$
$\Rightarrow$ relative virtual displacement has no component in the direction of the beam.

$\Rightarrow$ Consistent with the physics: particles cannot have a relative displacement along the beam

Moving Circular track

(2) Moving Circular Track



Constrained

$$[x - v(t - t_0)]^2 + y^2 - R^2 = 0$$

rheonomic holonomic constraint

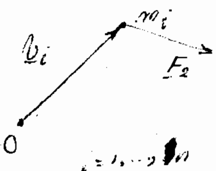
$$2[x - v(t - t_0)]\delta x + 2y\,\delta y = 0$$

NOTE: Virtual displacement are not true infinitesimal displacement in rheonomic Sys

$$2\,[x - V(t - t_s)]\,(dx - V\,dt) + 2y\,dy = 0$$

(dx, dy, dt) true infinitesimal displacement

(4) Virtual Work



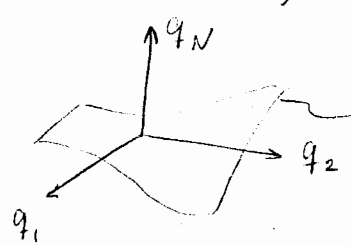
$$\delta W_{F_i} = \underline{F}_i \cdot \delta \underline{r}_i$$

Definition: A Constraint is Called ideal if The associated ideal force does zero virtual work

Consider specifically holonomic Constraints

$$f_j(q, t) = 0 \qquad q = \begin{Bmatrix} q_1 \\ \vdots \\ q_N \end{Bmatrix}$$

Space of q_i's are Called Configuration Space



$f_j(q, t) = 0$ Surface of dimension $N-1$
(instantaneous position of this Surface)

NOTE: the Constraint force $\underline{K}_j$ (Corresponding to $f_j(q,t)=0$ must be orthogonal to the above Surface)

$$\Rightarrow \underline{K}_j = \lambda \nabla f_j \qquad (\nabla \equiv \nabla_q) \qquad (1)$$

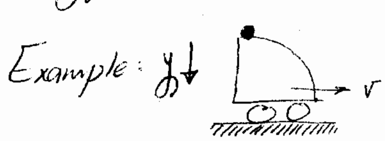
On the other hand, by definition, $\nabla f_j \cdot \delta q = 0 \quad (2)$

$$(1), (2) \longrightarrow \frac{1}{\lambda}\,\boxed{\underline{K}_j \cdot \delta q = 0}$$

$\hookrightarrow \delta W_{\underline{K}} = 0$

Any holonomic Constraint is ideal

Example:



$dW_N \neq 0$

because $f_j(q, t) = 0$

$$\underbrace{\nabla f_j \cdot dq}_{\frac{1}{\lambda} dW_N \neq 0} + \frac{\partial f}{\partial t}\,dt = 0$$

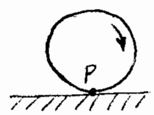
(because $\partial f / \partial t \neq 0$)

work done by the Constraint is non-zero but virtual work is zero (ideal Constraint)

General Result

true infinitesimal of the Constraint is only equal to the virtual work for sclronomic Constraint

(2) Rolling in 2D



$v_P = 0$ integrate

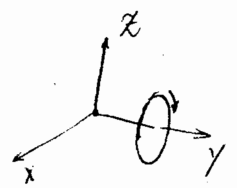
$x_P - R\varphi_P - (x^0_P - R\varphi^0_P) = 0$

$f(q, t) = 0$

ideal Constraint

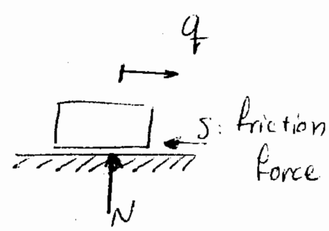
(3) 3D rolling

truly non holonomic



General argument given for holonomic Systems does not apply.
But One Can Still Show that this Constraint is ideal.

(4) Sliding over rough Surface



Scleronomic System

$dW_S = \delta W_S \neq 0$

$\Downarrow$

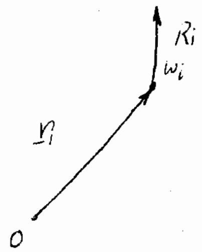
Constraint is not ideal

Trick: Consider S as an active force that is not part of the Constraint
$\Rightarrow$ Sliding becomes an ideal Constraint, and

$S = S(\mu, N; q, \dot{q}, \ldots)$

is just another active force

(5) Generalized Forces ≈ forces ~~expressed~~ acting in the direction of generaliz[ed] ~~Coordinate~~ Coordinate



$R_i = F_i + K_i$

Virtual work on ith particle

$\therefore \delta W^i = R_i \cdot \delta r_i = (F_i + K_i)\,\delta r_i$

Total virtual work does on system $\delta W = \sum_{i=1}^{n} F_i \cdot \delta r_i + \underbrace{\sum_{i=1}^{n} K_i\,\delta r_i}_{=0}$

$$= \sum_i F_i \cdot \sum_{j=1}^{N} \frac{\partial r_i}{\partial q_j}\,\delta q_j \qquad \text{if all constraints are ideal}$$

$$= \sum_{j=1}^{N} \underbrace{\left(\sum_{i=1}^{n} F_i \cdot \frac{\partial r_i}{\partial q_j} \right)}_{Q_j} \delta q_j$$

Q_j the generalized force associated with q_j

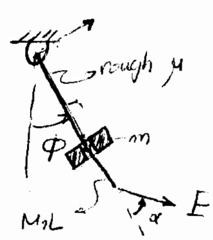
Special Case: generalized forces in the case when F_i's are all potential forces.

$$\delta W = \sum_i F_i\,\delta r_i = \sum_i \left(-\frac{\partial V}{\partial r_i}\right) \cdot \delta r_i$$

$$= -\delta V = \sum_{j=1}^{N} -\frac{\partial V}{\partial q_j}\,\delta q_j$$

$\Rightarrow$ for potential forces by definition $Q_j = -\frac{\partial V}{\partial q_j}$

Example: Sliding Collar on a rough beam (pendulum)
Subjected to a follower force



$\#DOF:\ 3+3-2-\underset{\text{holonomic}}{2} = 2$

$\Rightarrow$ 2 generalized coordinates

Question: Generalized forces Q_r & Q_φ

$$\delta W = \delta W^{potential} + \delta W^{nonpotential}$$

$$\delta W^{potential} = -\delta V = -\delta\left(Mg\frac{L}{2}\cos\varphi - mgr\cos\varphi\right)$$

$$= Mg\frac{L}{2}(-\sin\varphi)\,\delta\varphi + mgr(-\sin\varphi)\,\delta\varphi + mg\cos\varphi\,\delta r$$

$$Q_\varphi^{potential} = -\left(Mg\frac{L}{2} + mgr\right)\sin\varphi$$

$$Q_r^{potential} = mg\cos\varphi$$

Session 16

2-032

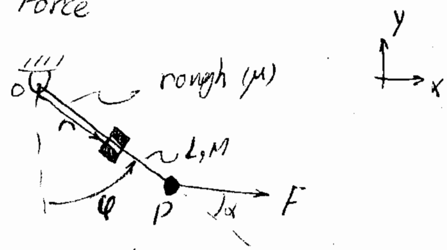
11/3-1

Analytic Mechanics
- generalized Coordinate
- Constraint
- Virtual disp
- Virtual work

$$\delta W = \underline{F} \cdot d\underline{r}$$

- Generalized Force

Example $g\downarrow$



point O is an ideal Constraint and by definition it doesn't do work.

$$\delta W = \underbrace{\delta W^{pot}}_{-\delta V} + \delta W^{non-pot}$$

$$\hookrightarrow Q_r^{pot} = mg\cos\varphi$$

$$Q_\varphi^{pot} = -(M\tfrac{L}{2} + mr)\, g\sin\varphi$$

$$\delta W^{non\text{-}pot} = \delta W_F + \delta W_{friction}$$

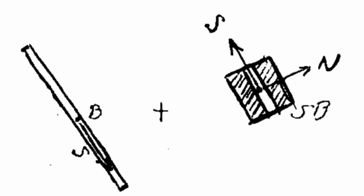
$$\delta W_F = \underline{F}\cdot\delta\underline{r}_P = F(\sin(\varphi+\alpha)\hat{i} - \cos(\varphi+\alpha)\hat{j}) \cdot \delta(L\sin\varphi\,\hat{i} - L\cos\varphi\,\hat{j})$$

$$= FL\sin\alpha\;\delta\varphi$$

$$\delta W_{friction} = \underbrace{\delta W^{beam}_{friction}}_{0} + \underbrace{\delta W^{collar}_{friction}}_{S\,\delta r}$$

virtual displacement is zero

FBD



$$S = -\mu|N|\,\mathrm{Sign}(\dot r)$$

what's $\underline{N}$?

Linear momentum principle Applied to Collar $\quad \dot{\underline{P}} = m\ddot{\underline{r}}_B = \underline{N} + m\underline{g} + \underline{S}$

$$(m\ddot{\underline{r}}_B)\cdot\underline{e}_N = N - mg\sin\varphi$$

$$\underline{r}_B = r\sin\varphi\,\hat{i} - r\cos\varphi\,\hat{j}$$

$$\ddot{\underline{r}}_B = (\ddot r\sin\varphi + 2\dot r\dot\varphi\cos\varphi + r\ddot\varphi\cos\varphi + r\dot\varphi^2\sin\varphi)\,\hat{i}$$
$$-(\ddot r\cos\varphi - 2\dot r\dot\varphi\sin\varphi - r\ddot\varphi\sin\varphi - r\dot\varphi^2\cos\varphi)\,\hat{j}$$

$$\underline{a}_B\cdot\underline{e}_N = 2\dot r\dot\varphi + r\ddot\varphi \implies N = m(r\ddot\varphi + 2\dot r\dot\varphi + g\sin\varphi) \implies S = -\mu m|r\ddot\varphi + 2\dot r\dot\varphi + g\sin\varphi|\,\mathrm{sign}$$

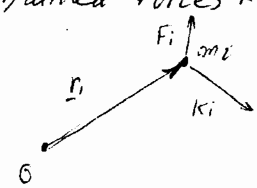
$$\delta W = \overbrace{\left[-g\sin\varphi\left(M\tfrac{L}{2}+mr\right)+F(\sin\alpha)\right]}^{Q_\varphi}\delta\varphi$$

$$+\underbrace{\left[mg\cos\varphi-\mu M|\ \ldots\ |\,\mathrm{sign}(\dot r)\right]}_{Q_r}\delta r$$

Now we turn to deriving eqn of motion using all these ingredients.

First let's formulate an extremum principle for the motion of a mechanical System

Consider a System of n particle $(m_1,\cdots,m_n)$ Subject to active forces F_i and Constrained forces K_i



assume Constraints are halonomic
even without this assumption we have

LMP: $\dot P_i = F_i + K_i \qquad \left(\sum_{i=1}^{n} \cdot\delta r_i\right)$

$$\Rightarrow \boxed{\sum_{i=1}^{n}(F_i-\dot P_i)\cdot\delta r_i = 0} \quad \text{D'Alembert's principle}$$

NOTE: if the System is at equilibrium $(\dot P_i = 0) \Rightarrow \boxed{\sum F_i\cdot\delta \vec r_i = 0}$ principle of Virtual work

NOTE:
- $\sum_{i=1}^{n} F_i\cdot\delta r_i = \delta W$
- $\sum_{i=1}^{n}\dot P_i\cdot\delta r_i = \sum_i m_i\ddot r_i\cdot\delta r_i = \sum m_i\left[\frac{d}{dt}[\dot r_i\cdot\delta r_i] - \dot r_i\,\delta\dot r_i\right]$

$$= \sum_i m_i\left[\frac{d}{dt}(\dot r_i\cdot\delta r_i)\right] - \delta\underbrace{\sum_i \tfrac{1}{2}m_i(\dot r_i\cdot\dot r_i)}_{T}$$

$$\Rightarrow \delta T + \delta W = \sum_{i=1}^{n} m_i\frac{d}{dt}[\dot r_i\cdot\delta r_i]$$

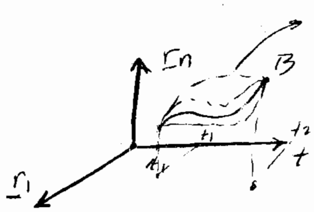
Integrate along the motion $(r_1(t),\ldots,r_n(t))$

$$\int_{t_1}^{t_2}\delta(T+W)\Big|_{r(t)}dt = \sum_i\int_{t_1}^{t_2} m_i\frac{d}{dt}[\dot r_i\,\delta r_i]\,dt$$

Variation

kinematically admissible path between A & B
Satisfy the Constraints AND Connect A & B
in the extended Configuration Space at $(r_1,\ldots,r_n)$

Geometry



Initial position A: $r_1(t_1), \cdots, r_n(t_1)$
Final position B: $r_1(t_2), \cdots, r_n(t_2)$

$\Rightarrow \delta r_i \big|_{t=t_1} = 0 \qquad \delta r_i \big|_{t=t_2} = 0$

$$\Rightarrow \int_{t_1}^{t_2} \delta(T+W)\Big|_{r(t)} dt = \sum_i m_i \left[\dot{r}_2 \cdot \delta r_2 \right]_{t_1}^{t_2} = 0$$

$$\Rightarrow \boxed{\int_{t_1}^{t_2} \delta(T+W)\Big|_{\underline{r}(t)} dt = 0}$$ Extended hamilton Principle (*)

Assume all forces are (active forces) are potential forces

$$\delta W = -\delta V$$

then define the Lagrangian $L = T - V$

$\Rightarrow$ (*) gives $\boxed{\delta \int_{t_1}^{t_2} L(\underline{r}(t), \dot{\underline{r}}(t))\, dt = 0}$ principle of least action (Hamilton's principle)

In other words The function

$$I = \int_{t_1}^{t_2} L(\vec{r}_i(t), \dot{\vec{r}}_i(t); t)\, dt$$

defined for any kinematically admissible path admites an extremum along the actual motion of mechanical system. $\boxed{\delta I = 0}$ I is called the action

Analogy $g(x_1, \cdots, x_n)$ [redacted] (function of n variables)

At points of extremal $dg = 0$, indeed $dg = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i \equiv 0$

$$\Leftrightarrow \frac{\partial g}{\partial x_i} = 0 \qquad i = 1, \cdots, n$$

Session 17

For systems w/ ideal constraints $\int_{t_1}^{t_2} (\delta T + \delta W)\Big|_{\underline{r}(t)} dt = 0$

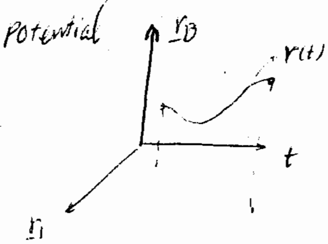
if in addition, all forces are potential

$\boxed{\delta I = 0}$

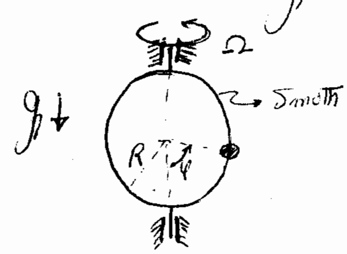
action

$I = \int_{t_1}^{t_2} L(\underline{r}(t), \dot{\underline{r}}(t)) dt$

$L = T - V$

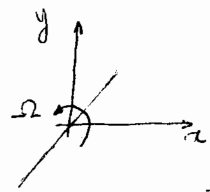


Example: Bead moving on rotatig ring



- Active forces: gravity (potential)
- Constraints:

(1)



$\frac{y}{x} = \tan \Omega t$

i.e. $y \cos \Omega t - x \sin \Omega t = 0$

(2) $x^2 + y^2 + z^2 - R^2 = 0$

2 holonomic constraints

$\Rightarrow$ System is holonomic

Principle of least action applies $\delta I = 0$

$\# DOF = 3 - 2 = 1$

$L = T - V = \frac{1}{2} m |\underline{v}|^2 + (+mgR \cos \varphi)$

$= \frac{1}{2} m ((R \sin \varphi \, \Omega)^2 + R^2 \dot{\varphi}^2) + mgR \cos \varphi$

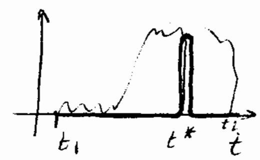
$\delta I = \delta \int_{t_1}^{t_2} [\frac{1}{2} m (R^2 \sin^2 \varphi \, \Omega^2 + R^2 \dot{\varphi}^2) + mgR \cos \varphi] dt$

$= \int_{t_1}^{t_2} [\frac{1}{2} m (R^2 \Omega^2 2 \sin \varphi \cos \varphi \, \delta \varphi + R^2 2 \dot{\varphi} \delta \dot{\varphi}) - mgR \sin \varphi \, \delta \varphi] dt$

NOTE: $\int_{t_1}^{t_2} (\overset{u}{\dot{\varphi}} \, \overset{v'}{\delta \dot{\varphi}}) dt = [\dot{\varphi} \, \delta \varphi]_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\varphi} \, \delta \varphi \, dt$

$\delta I = [\frac{1}{2} m (R^2 \Omega^2 \sin 2\varphi \mp mR^2 \ddot{\varphi}) - mgR \sin \varphi] \delta \varphi \, dt = 0$

_ must vanish for all t_1 and t_2 _ for kinematically admissible $\delta \varphi$



is [ ] = 0 for all t?
assume not $\Rightarrow$ $[\;\;]\big|_{t=t^*} \neq 0$ for some t^*

$\Rightarrow \delta I \neq 0$ ↯

$\Rightarrow mR^2\ddot{\varphi} + mgR\sin\varphi - \frac{1}{2}mR^2\sin 2\varphi = 0$

NOTE: Constraint force did not enter calculation

the above calculation can be performed for general systems with ideal holonomic constraints
Extended Hamilton's Principle

$$\int_{t_1}^{t_2}(\delta T + \delta W)\Big|_{r(t)} dt = \int_{t_1}^{t_2}\Big[\delta T + \big(\underbrace{-\delta V}_{\text{potential force}} + \underbrace{\sum_{j=1}^{N} Q_j\,\delta q_j}_{\text{non-potential}} + \delta W^{(\text{Const})}\big)\Big]dt$$

$$= \int_{t_1}^{t_2}\Big(\underset{\uparrow\, L(q,\dot{q},t)}{\delta L} + \sum_{j=1}^{N} Q_j\,\delta q_j\Big)dt = \sum_{j=1}^{N}\int_{t_1}^{t_2}\Big[\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta\dot{q}_j + Q_j\,\delta q_j\Big]dt$$

NOTE: $$\int_{t_1}^{t_2}\Big(\frac{\partial L}{\partial \dot{q}}\,\delta\dot{q}_j\Big)dt = \overset{0}{\Big[\frac{\partial L}{\partial \dot{q}_j}\,\delta q_j\Big]_{t_1}^{t_2}} - \int_{t_1}^{t_2}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\,\delta q_j\,dt$$

$$\sum_{j=1}^{N}\int_{t_1}^{t_2}\Big[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + Q_j\Big]\delta q_j\,dt = 0$$

This must hold for all t_1, t_2
for all δq_j, $j=1,\dots,N$

For any j, repeat argument from the previous example, setting
$\delta q_1, \delta q_2, \dots, \delta q_{j-1}, \delta q_{j+1}, \dots, \delta q_N = 0$

Here we use heavily the independence of q_j's (System is holonomic)

$$\Rightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j} \quad j=1,\dots,N \qquad N: \#DOF$$

$L = T - V$

Analytical Mechanics

For holonomic systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$L = T - V; \quad \delta W = \sum Q_i \delta q_i$$

Finding constraint forces using the lagrangian approach

- Consider $q_1, \dots, q_n$ complete but not independent set of coordinate they satisfy some holonomic constraint whose constraint forces we seek
- Assume that we have m holonomic ~~coordinates~~ constraints satisfied by these coordinates

$$\sum_{j=1}^{n} a_{ij}\, dq_j + b_i\, dt = 0 \qquad i = 1, \dots, m$$

Select scalars $\lambda_i,\ i = 1, \dots, m$

$$(1) \Rightarrow \sum_{j=1}^{n} a_{ij}\, \delta q_j = 0 \quad i = 1, \dots, m \quad \Rightarrow \quad \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m} a_{ij}\, \delta q_j = 0 \qquad (2)$$

By extending hamilton principle

$$(3) \qquad \int_{t_1}^{t_2} (\delta T + \delta V)\Big|_{r(t)} dt = 0$$

Integrate (2) along $r(t)$, add to (3):

$$\int_{t_1}^{t_2} \left(\delta T + \delta W + \sum \lambda_i a_{ij}\, \delta q_j\right)\Big|_{r(t)} dt = 0$$

repeat argument leading to lagrange's eqs of motion (except for the last step) to obtain:

$$\sum_{j=1}^{n} \int_{t_1}^{t_2} \left[-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial q_j} + Q_j + \sum_{i=1}^{m} \lambda_i a_{ij} \right] \delta q_j\, dt = 0$$

Idea: is making [ ]$_j$ vanish for all j, use n-m independent δq_j's, AND select $\lambda_1, \dots, \lambda_m$ in a fashion so that the remaining in brackets vanish

$$\Rightarrow \boxed{\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j + \sum_{i=1}^{m} \lambda_i a_{ij} \qquad j = 1, \dots, n \\ &\text{Add: } \sum_{j=1}^{n} a_{ij}\, \dot{q}_j + b_i = 0 \qquad i = 1, \dots, m \end{aligned}}$$

n+m eqs

n+m unknowns

λ_i: Lagrangian multipliers

$$\sum^{n} a_{ij}\, \dot{q}_j + b_i = 0 \qquad i = 1, \dots, m$$

NOTE:

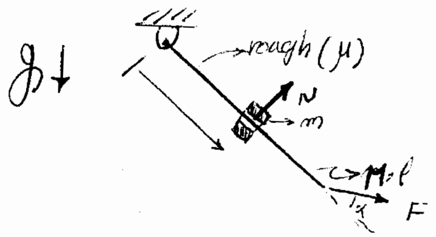
$K_j = \sum_i \lambda_i a_{ij}$

is the j^{th} Coordinate resultant force

Also The above formulation Covers non-holonomic System as well, because Constraints Can also be written in the form (1)

Example

Reconsider "Collar sliding on pendulum under the effect of follower force"



Question: Constraint force N ?

. Select: $q_1 = r$, $q_2 = \varphi$ } determine position of center of mass of Collar

$q_3 = \theta$ angle of beam with horizontal

$\Rightarrow n = 3$

. Constraint: $q_2 - q_3 = 0$ $(m=1)$ $(a_{11} q_1 + a_{12} q_2 + a_{13} q_3 = 0)$

$a_{11} = 0;$ $a_{12} = 1;$ $a_{13} = -1$

. Active generalized forces unrelated to Constraints (non-potential)

in q_1 direction $Q_1 = S$

$Q_2 = 0$

$Q_3 = F\ell \sin\alpha$

, $m = 1 \Rightarrow$ Only one lagrangian multipliers λ_i

$L = T - V$

$T = T_{beam} + T_{Collar} = \frac{1}{6} M\ell^2\dot{\theta}^2 + \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2)$

$V = V_{beam} + V_{Collar} = -Mg\frac{\ell}{2}\cos\theta - mgr\cos\varphi.$

$L = \frac{1}{2} M\ell^2\dot{\theta}^2 + \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) + Mg\frac{\ell}{2}\cos\theta + mgr\cos\varphi$

Equation of motion: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = S + \lambda_1 \underbrace{a_{11}}_{=0}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = \lambda_1 a_{12} = \lambda_1$ K_2

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = F\ell\sin\alpha + \lambda_1 a_{13}$
$= F\ell\sin\alpha - \lambda_1$

$K_1 = \lambda_1 = \frac{d}{dt}(mr^2\dot{\varphi}) + mgr\sin\varphi$

$= 2m\dot{r}r\dot{\varphi} + mr^2\ddot{\varphi} + mgr\sin\varphi$

To obtain N: $\delta W^{non\text{-}potential} = \underbrace{(S)}_{friction}\delta r + \underbrace{(\quad F\ell\sin\alpha - Nr)}_{Follower\ Force}\delta\theta + Nr\,\delta\varphi$ K_j

$$N = \frac{1}{r} K_2$$

$$= 2m\dot{r}\dot{\varphi} + mr\ddot{\varphi} + mg\sin\varphi$$

Session 19

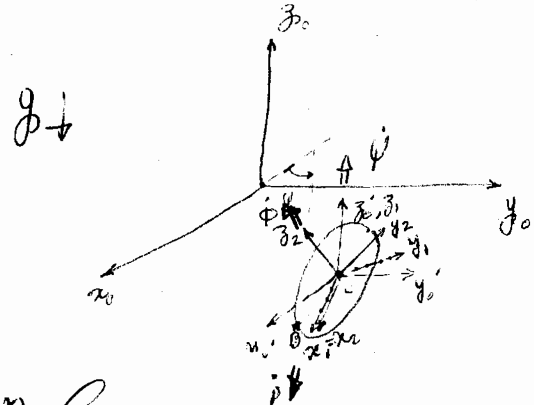
Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_j} = Q_j + \sum_{i=1}^{m} \lambda_i a_{ij} \qquad (1)$$

$i = 1, \dots, m$
$j = 1, \dots, n$

$$\sum_{j=1}^{n} a_{ij} \dot q_j + b_i = 0$$

Example Use of lagrangian multipliers for nonholonomic systems
rolling penny:
Thin disk rolling without slip on a horizontal plane



Initial choice of Coordinates
$(\underbrace{x, y, z}_{\text{position of } c}, \underbrace{\psi, \nu, \varphi}_{\text{Euler angles of "3-1" type}})$

(*) Constraints: $\underline{v}_B = \underline{0} \Rightarrow$ 3 Constraints $\Rightarrow$ # DOF = 6 - 3 = 3

$\underline{v}_B = \underline{v}_c + \underline{\omega} \times \underline{r}_{cB}$; $\underline{v}_c = \dot x\, \underline{i} + \dot y\, \underline{j} + \dot z\, \underline{k}$, (i, j, k are unit vectors in x_0', y_0', z_0' frame)

$$\omega = \dot\psi + \dot\nu + \dot\varphi$$
$$= \dot\nu\, \hat i_2 + \dot\psi \sin\nu\, \hat j_2 + (\dot\psi \cos\nu + \dot\varphi)\hat k_2$$

Here (i_2, j_2, k_2) are unit vectors in the (x_2, y_2, z_2) frame)

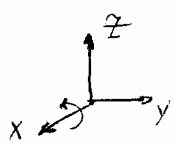
Also $\underline{r}_{cB} = -R\, j_2$

$\underline{\omega} \times \underline{r}_{cB} = R(\dot\psi \cos\nu + \dot\varphi)\, i_2 - R\dot\nu\, k_2$

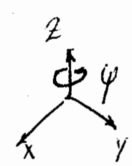
NOTE: $\underline{i}_2 = \underbrace{\underline{\underline{R}}_3\, \underline{\underline{R}}_1}_{} \underline{\underline{R}}_3^T \underline{\underline{R}}_3\, \underline{i} = \underline{\underline{R}}_3\, \underline{\underline{R}}_1\, \underline{i}$

Representation of "1" rotation in the $[i_1, j_1, k_1]$ frame

where $\underline{\underline{R}}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\nu & -\sin\nu \\ 0 & \sin\nu & \cos\nu \end{pmatrix}$



$\underline{\underline{R}}_3 = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$$i_2 = \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \end{pmatrix} ; \quad K_2 = \begin{pmatrix} \sin\psi \sin\nu \\ -\cos\psi \sin\nu \\ \cos\nu \end{pmatrix} \qquad (\underline{K}_2 = \underline{B}_3 \underline{B}_1 \underline{k})$$

(*) gives $i=1$ (a) $\dot{x} + \dot\psi R\cos\psi\cos\nu + \dot\varphi R\cos\psi - \dot\nu R\sin\psi\sin\nu = 0$

$i=2$ (b) $\dot{y} + \dot\psi R\sin\psi\cos\nu + \dot\varphi R\sin\psi - \dot\nu R\cos\psi\sin\nu = 0$

} non halonomic

(c) $\dot{z} - \dot\nu R\cos\nu = 0 \longrightarrow \boxed{z - R\sin\nu = 0}$ Halonomic

halonomic Constraint makes it possible to pass to the generalized Coordinates

$$\begin{matrix}(x, & y, & \psi, & \nu, & \varphi) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ q_1 & q_2 & q_3 & q_4 & q_5\end{matrix} \qquad (5)$$

Since the rolling Constraint is ideal eq (1) applies

$$L = T - V$$

$$T = \frac{1}{2} m \, |V_c|^2 + \frac{1}{2} \underline{\omega}^T \underline{\underline{I}}_c \underline{\omega}$$

$$= \frac{1}{2} m (\dot x^2 + \dot y^2 + \dot z^2) + \frac{1}{2} \left[I_{x2} \dot\nu^2 + I_{y2} \dot\psi^2 \sin^2\nu + I_{z2} (\dot\psi\cos\nu + \dot\varphi)^2 \right]$$

USE HOLONOMIC CONSTRAINT

$$I_{x2} = I_{y2} = \frac{1}{4} m R^2 \qquad I_{z2} = \frac{1}{2} m R^2$$

$$= \frac{1}{2} m (\dot x^2 + \dot y^2 + \dot\nu^2 R^2 \cos^2\nu) + \frac{1}{8} m R^2 \left[\dot\nu^2 + \dot\psi^2 \sin^2\nu + 2(\dot\psi\cos\nu + \dot\varphi)^2 \right]$$

$$V = mgR\sin\nu$$

$$\Rightarrow L = \frac{1}{2} m (\dot x^2 + \dot y^2 + \dot\nu^2 R^2 \cos^2\nu) + \frac{1}{8} m R^2 \left[\dot\nu^2 + \dot\psi^2 \sin^2\nu + 2(\dot\psi\cos\nu + \dot\varphi)^2 \right] - mgR\sin\nu = 0$$

- $Q_j = 0. \quad j = 1, \dots, 5$

(No non-potential active forces)

Identify "a_{ij}": $\quad i = 1, \dots, 2 \quad j = 1, \dots, 5$

$a_{11} = 1 \quad a_{12} = 0 \quad a_{13} = R\cos\psi\cos\nu \quad a_{14} = -R\sin\psi\sin\nu$

$a_{15} = R\sin\psi$

$a_{21} = 0 \quad a_{22} = 1 \quad a_{23} = R\sin\psi\cos\nu$

$a_{24} = R\cos\psi\sin\nu \quad a_{25} = R\sin\psi$

Eq. of motion

Cyclic Coordinate

(1) $m\ddot{x} = \lambda_1$

(2) $m\ddot{y} = \lambda_2$

no dependence on ψ, but since the problem is non-holonomic it doesn't help us

(3) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \underbrace{\frac{\partial L}{\partial \psi}}_{0} = R\cos\nu(\lambda_1 \cos\psi - \lambda_2 \sin\psi)$

(4) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\nu}}\right) - \frac{\partial L}{\partial \nu} = R\sin\nu(-\lambda_1 \sin\psi + \lambda_2 \cos\psi)$

(5) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = R(\lambda_1 \cos\psi + \lambda_2 \sin\psi)$

+ Constraints (a) & (b)

In Solving eqs, use (1) and (2) to eliminate λ_1 & λ_2 from the remaining equations.

$\Rightarrow$ 5 ODE for five Coordinates

Session 20

Equilibria and their Stability

Assume (1) System is holonomic scleronomic $\Rightarrow L(q,\dot{q})$, $q\ (q_1, \dots, q_m)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q(q,\dot{q}) \qquad \text{assumed } \frac{\partial Q}{\partial t} = 0$$

2) equilibrium or fixed point : $q = q^0 = \text{Const}$ $(\Rightarrow \dot{q} = 0)$

$\Rightarrow$ all functions of q & $\dot{q}$ are Constant in time at equilibria

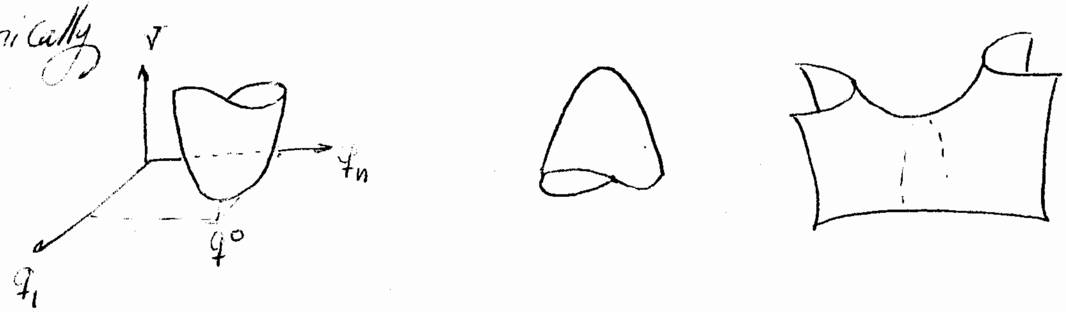
$$-\frac{\partial L}{\partial q}(q,\dot{q}) = Q(q^0, 0)$$

Stability in systems with potential forces

$$Q = 0 \qquad -\frac{\partial L}{\partial q}(q^0, 0) = -\frac{\partial}{\partial q}(T - V)\Big|_{\substack{q=q^0 \\ \dot{q}=0}} = \frac{\partial V}{\partial q}(q^0) = 0$$

holds at equilibria for Conservative systems

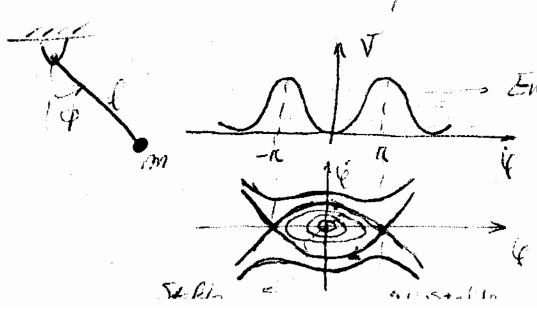
Geometrically



Stability: q^0 is stable if for all small perturbations the resulting motion stays close to q^0 $\quad \forall \varepsilon > 0\ \exists \delta > 0$ Such that for all $|q(0) - q^0| \leq \delta$ we have $|q(t) - q^0| < \varepsilon$ for all $t \geq 0$

q^0 is unstable if not stable, (i.e., there is at least one perturbation that grows)

Example

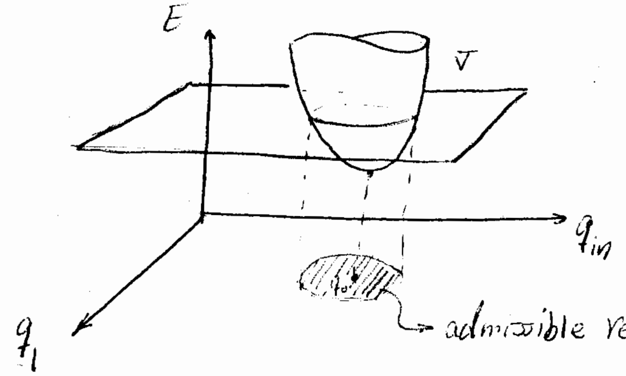


$\rightarrow$ Energy levels $E = T + V$

Stable = unstable

Stability Criterion (Dirichlet) in a Conservative System

an equilibrium q° is stable if and only if V has a ~~[illegible]~~ (strict) local minimum.



$E = T + V = \text{Const}$

$T \geq 0$

E is slightly over $V(q^{\circ})$

→ admissible region for trajectory $q(t)$ with q° in R

How do we find local minimum of V

- $\left.\dfrac{\partial V}{\partial q}\right|_{q=q^{\circ}} = 0$ (extremum point)
- $\left.\left[\dfrac{\partial^2 V}{\partial q_i \, \partial q_j}\right]\right|_{q=q^{\circ}}$ is positive definite

(Hessian matrix)

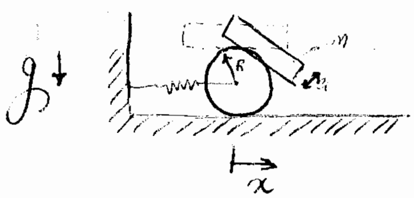
Sufficient and necessary condition for the positive definiteness of a ~~system~~ symmetric matrix:

1) all its eigenvalues are positive

2) $\begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \end{bmatrix}$ $D_1 = \det[a_{11}]$ $D_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\det(D_i) > 0$ $i = 1, \dots, n$

Example (written dynamics qual question 2004)

Rolling disk with tipping block, Constrained by spring



both objects roll without slip

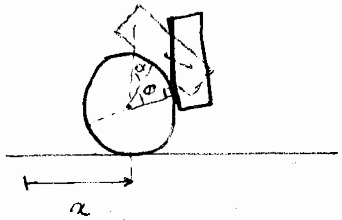
$\# DOF = 2 \times 3 - 2 \times 2 = 2$

- active forces are potential (gravity - spring)
- Constraint forces do not do work (rolling)

→ System is Conservative
⇓
Dirichlet theorem applies

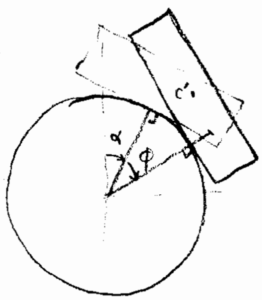
Understand displacement of block by

- first rolling the disk to positive x with the block fixed to the disk
- then superimpose the rolling of the disk to positive φ



$$\alpha = \frac{x}{R}$$

$$\sin(\alpha+\varphi)(R\varphi)$$
$$\cos(\alpha+\varphi)(R+\tfrac{a}{2})$$
$$y_c = R\varphi \sin(\tfrac{x}{R}+\varphi) + (R+\tfrac{a}{2})\cos(\tfrac{x}{R}+\varphi)$$



$$V = V^{\text{spring}} + V^{\text{block}} + V^{\text{disk}}$$
$$= \frac{1}{2}kx^2 + mg\left[(R+\tfrac{a}{2})\cos(\tfrac{x}{R}+\varphi) + R\varphi\sin(\tfrac{x}{R}+\varphi)\right]$$

$$\begin{cases} \dfrac{\partial V}{\partial x} = kx + mg\left[\dfrac{2R+a}{2R}\sin(\tfrac{x}{R}+\varphi) + \varphi\cos(\tfrac{x}{R}+\varphi)\right] \\ \dfrac{\partial V}{\partial \varphi} = mg\left[-\dfrac{a}{2}\sin(\tfrac{x}{R}+\varphi) + R\varphi\cos(\tfrac{x}{R}+\varphi)\right] \end{cases}$$

Note: $\dfrac{\partial V}{\partial x}^{(0,0)} = 0$

$\dfrac{\partial V}{\partial \varphi}^{(0,0)} = 0$

$\Rightarrow (x,\varphi) = (0,0)$ is indeed equilibrium

Stability criterion (Dirichlet) in a conservative system

For stability: $\dfrac{\partial^2 V}{\partial x^2} = k + mg\left[\dfrac{2R+a}{2R^2}\cos(\tfrac{x}{R}+\varphi) - \dfrac{\varphi}{R}\sin(\tfrac{x}{R}+\varphi)\right]$

$$\frac{\partial^2 V}{\partial x \partial \varphi} = mg\left[-\frac{a}{2R}\cos(\tfrac{x}{R}+\varphi) - \varphi\sin(\tfrac{x}{R}+\varphi)\right]$$

$$\frac{\partial^2 V}{\partial \varphi^2} = mg\left[\frac{2R-a}{2}\cos(\tfrac{x}{R}+\varphi) - R\varphi\sin(\tfrac{x}{R}+\varphi)\right]$$

Hessian matrix of V at equil. $D^2V = \begin{bmatrix} k - mg\frac{2R+a}{2R^2} & -mg\frac{a}{2R} \\ \text{Sym.} & mg\frac{2R-a}{2} \end{bmatrix}$

D^2V is positive definition if and only if (1) $k > mg\dfrac{2R+a}{2R^2}$

(2) $\left[k - mg\frac{2R+a}{2R^2}\right] mg\frac{2R-a}{2} > \frac{m^2g^2a^2}{\ldots}$

if (1) holds, (2) can only hold if $R > a/2$, in which case (2) simplifies

(3) $$K > mg\left[\frac{2R+a}{2R^2} + \frac{a^2}{2R^2(2R-a)}\right] = \frac{2mg}{2R-a}$$

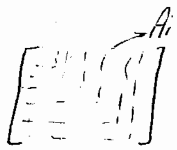
thus (3) is stronger than (1), the final set of conditions for stability

$$R > \frac{a}{2}, \quad K > \frac{2mg}{2R-a}$$

Stability of equilibrium

For systems with Conservative forces $\quad \frac{\partial V(q_0)}{\partial q} = 0 \qquad (1)$

$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q_0}$ pos. definite



$\det(A_i) > 0$
$i = 1, \dots, n$

Stability & Small Oscillations in general holonomic Systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \underline{Q} \qquad q = (q_1, \dots, q_N)$$

$$L = T - V \qquad \underline{Q}(q, \dot{q}, t)$$

Kinetic Energy:

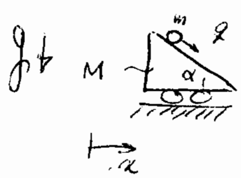
$$T = \frac{1}{2} \sum_{i=1}^{n} m_i |\dot{\underline{r}}_i|^2; \quad \underline{r}_i = \underline{r}_i(q_1, \dots, q_N, t)$$

$$\Rightarrow T = \frac{1}{2} \sum_{i=1}^{n} m_i \left[\sum_{j=1}^{N} \left(\frac{\partial \underline{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \underline{r}_i}{\partial t} \right) \right] \cdot \left[\qquad \right]$$

$$T = \underbrace{\frac{1}{2} \sum_{i,j=1}^{N} m_{ij}(q, t) \dot{q}_i \dot{q}_j}_{T_2} + \underbrace{\sum_{i=1}^{N} b_i(q, t) \dot{q}_i}_{T_1} + \underbrace{c(q, t)}_{T_0}$$

Natural Mechanical System: By definition is one for which $T_1 = T_0 = 0$

Example (1)



$$T = \frac{1}{2} m \left[(\dot{x} + \dot{q}\cos\alpha)^2 + (\dot{q}\sin\alpha)^2 \right] + \frac{1}{2} M \dot{x}^2$$

Gen. Coordinates $(x, q) \Rightarrow$ Natural System

(2) Same Problem with the Constraint

$\left.\begin{array}{l} \dot{x} = V(t) = \text{prescribed} \\ \text{(Integrable non holonomic Constraint)} \end{array}\right\} \Rightarrow$ gen. Coordinate q

$$T = \frac{1}{2} m \left[(V + \dot{q}\cos\alpha)^2 + (\dot{q}\sin\alpha)^2 \right] + \frac{1}{2} M V^2$$

non-natural System

(3) if $q = (\hat{q}, \varphi)$, $\frac{\partial L}{\partial \varphi} = 0$; $Q_\varphi = 0 \longrightarrow \varphi$ is a cyclic (ignorable) coordinate

$\Rightarrow \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot\varphi}\right)}_{P_p} = 0$ $\Rightarrow P_p = \frac{\partial L}{\partial \dot\varphi} \Rightarrow \dot\varphi = F(P, \hat{q}, \dot{\hat{q}})$

P_p : gen. momentum associated with φ

$\Downarrow$

reduced set of eqn for $\hat{q}$

$\Rightarrow$ Kinetic energy in reduced coordinates

(Assume unreduced system natural)

$$T = \frac{1}{2}\sum_{i,j=1}^{N-1} m_{ij}\dot{\hat{q}}_i\dot{\hat{q}}_j + \frac{1}{2}\sum_{i=1}^{N-1} m_{iN}\dot{\hat{q}}_i\underbrace{\dot\varphi}_{F(\varphi,\hat{q},\dot{\hat{q}})} + \frac{1}{2}m_{NN}\underbrace{\dot\varphi^2}_{F(\varphi,\hat{q},\dot{\hat{q}})}$$

$\Rightarrow$ T_1 & T_2-type term

$\Rightarrow$ reduced system for $\hat{q}$ is non-natural

For general non-natural system:

$$L = T - V = T_2 + T_1 + T_0 - V$$

let: $T_2 = \frac{1}{2}\dot{q}^T \underline{\underline{M}}(q,t)\dot{q}$, $\underline{\underline{M}} = [m_{ij}]_{i,j=1}^{n}$

$T_1 = \underline{b}.\dot{q} = \underline{b}^T(q,t)\dot{q}$; $\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$

$V = V(q,t)$

$Q = Q(q, \dot{q}, t)$

(Assuming $\frac{\partial T_0}{\partial q} = 0$)

$\Rightarrow$ Lagrangian eq. of motion $\frac{d}{dt}(\underline{\underline{M}}\dot{q} + \underline{b}) - \underbrace{\frac{1}{2}\dot{q}^T\frac{\partial \underline{\underline{M}}}{\partial q}\dot{q}}_{\text{Nonlinear term}} - \frac{\partial \underline{b}^T}{\partial q}\dot{q} + \frac{\partial V}{\partial q} = \underline{Q}$

Assume $\frac{\partial \underline{\underline{M}}}{\partial t} = \underline{0}$, $\frac{\partial \underline{b}}{\partial t} = \underline{0}$

$$\Rightarrow \underline{M}\ddot{q} + \frac{1}{2}\dot{q}^T\frac{\partial \underline{\underline{M}}}{\partial q}\dot{q} + \left[\frac{\partial b}{\partial q} - \frac{\partial b^T}{\partial q}\right]\dot{q} + \frac{\partial V}{\partial q} - \frac{\partial T_0}{\partial q} = \underline{Q}(q,\dot{q}) \quad (*)$$

equilibria: $q = q_0 \Rightarrow \dot{q} = 0,\ \ddot{q} = 0$ — we can assume $\frac{\partial T_0}{\partial t} \neq 0$!!

$\left.\frac{\partial V}{\partial q}\right|_{q_0} = Q(q_0, 0, t)$ defines equilibria

to understand the stability of q_0, linearize $(*)$ about q_0

$$\Rightarrow \underline{M}(q_0)\ddot{q} + \left[\frac{\partial b}{\partial q} - \frac{\partial b^T}{\partial q}\right]\Bigg|_{q=q_0}\dot{q} + \cancel{\left.\frac{\partial V}{\partial q}\right|_{q_0}} + \left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0}(q-q_0) + O(2)$$
$$- \cancel{\underline{Q}(q_0,0,t)} + \left.\frac{\partial Q}{\partial q}\right|_{q=q_0}(q-q_0)$$
$$+ \left.\frac{\partial Q}{\partial \dot{q}}\right|_{q_0, q}\dot{q} + O(2) = 0$$

let (1) $\underline{\underline{M}} = \underline{\underline{M}}(q_0)$: mass matrix

Symmetric ($\underline{\underline{M}} = \underline{\underline{M}}^T$)
and positive-definite ($\underline{X}^T \underline{\underline{M}} \underline{X} > 0$)
$\underline{X} \neq 0$

(2) $\underline{\underline{G}} = \left[\frac{\partial b}{\partial q} - \frac{\partial b^T}{\partial q} \right] \Big|_{q=0}$ gyroscopic matrix
only presents for non-natural system

$\underline{\underline{G}}^T = -\underline{\underline{G}}$ skew-symmetric

(3) $\underline{\underline{K}} = \frac{\partial^2 V}{\partial q^2} \Big|_{q=q_0}$ stiffness matrix $\quad \underline{\underline{K}} = \underline{\underline{K}}^T \Rightarrow K$ is symmetric
if q is stable eq. $\Rightarrow K$ = positive definite

(4) $\underline{\underline{B}} = -\frac{\partial Q}{\partial q} \Big|_{q=q_0}$

$\underline{\underline{C}} = -\frac{\partial Q}{\partial \dot{q}} \Big|_{q=q_0}$

$\Rightarrow$ Linearized eq. of motion
with $\underline{x} = q - q_0$

$$\underline{\underline{M}} \cdot \ddot{\underline{x}} + (\underline{\underline{G}} + \underline{\underline{C}}) \dot{\underline{x}} + (\underline{\underline{K}} + \underline{\underline{B}}) \underline{x} = \underline{0}$$

Session 22

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Linearized eq of motion for holonomic systems

$\underline{x} = \underline{q} - \underline{q}_0$, $\underline{\underline{M}}\,\ddot{\underline{x}} + (\underline{\underline{G}} + \underline{\underline{C}})\dot{\underline{x}} + (\underline{\underline{K}} + \underline{\underline{B}})\underline{x} = \underline{0}$

$-\frac{\partial V}{\partial q}\Big|_{q_0} = Q\big|_{q_0}$

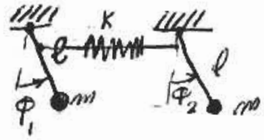
In the Conservative & natural Case
(potential)

$$\underline{\underline{M}}\,\ddot{\underline{x}} + \underline{\underline{K}}\,\underline{x} = \underline{0} \quad (1)$$

Simplest derivation from quadratic lagrangian

$$L = \frac{1}{2}\dot{\underline{x}}^T \underline{\underline{M}}\,\dot{\underline{x}} - \frac{1}{2}\underline{x}^T \underline{\underline{K}}\,\underline{x} + O(\varepsilon)$$

Example



Spring is unstretched at $\varphi_1 = \varphi_2 = 0$
Assume Unstretched length $= \underline{0}$

$T = \frac{1}{2}ml^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2)$; Equilibrium. $\varphi_{10} = \varphi_{20} = 0 \Rightarrow \underline{x} = \underline{\varphi} - \underline{\varphi}_0 = \underline{\varphi}$

$= \frac{1}{2}ml^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2)$

Session 22

$$V = mgl(1-\cos\varphi_1) + mgl(1-\cos\varphi_2)$$
$$+ \frac{1}{2}k\left[(h\cos\varphi_1 - h\cos\varphi_2)^2 + (h\sin\varphi_1 - h\sin\varphi_2)^2\right]$$

$$\Rightarrow L = \frac{1}{2}ml^2(\dot{x}_1^2+\dot{x}_2^2) - mgl\left[1-(1-\frac{\varphi_1^2}{2}+\cdots)+1-(1-\frac{\varphi_2^2}{2}+\cdots)\right]$$
$$-\frac{1}{2}kh^2\Big[\underbrace{(1-\frac{\varphi_1^2}{2}+\cdots-1+\frac{\varphi_2^2}{2}+\cdots)^2}_{O(4)}+(\varphi_1-\varphi_2+\cdots)^2\Big]$$

$$= \frac{1}{2}ml^2(\dot{x}_1^2+\dot{x}_2^2)$$
$$-\frac{1}{2}mgl(x_1^2+x_2^2) - \frac{1}{2}kh^2(x_1-x_2)^2 + O(3)$$

Sym, pos. def. $\quad$ Sym

$$\Rightarrow \underline{\underline{M}} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}; \quad \underline{\underline{K}} = \begin{pmatrix} mgl+kh^2 & -kh^2 \\ -kh^2 & mgl+kh^2 \end{pmatrix} \rightarrow \text{Stiffness}$$

(mass matrix)

if we expand about any other equilibrium point, we'll get different matrices

Since the equilibrium is stable $\rightarrow$ $\underline{K}$ has to be positive definite and it is

Systematic Solution of (1):

Trial Solution $\tilde{x}(t) = x_0 e^{\lambda t}$; general Solution $x(t) = \sum c_i \tilde{x}_i(t)$

$\underline{x}_0 \neq \underline{0}$ $\qquad$ From initial Conditions

plug into (1):

(2) $\quad (\lambda^2 \underline{\underline{M}} + \underline{\underline{K}})\underset{a}{\underline{x}_0} = \underline{0}$, let $\underline{x}_0 = \underline{a}$

Note: if a is a Solution, so is $c\underline{a}$ $\Rightarrow$ mode shapes are never unique in magnitude

- left multiply (2) by $\underline{a}^T$ $\Rightarrow$ $\boxed{\lambda^2 = -\frac{\underline{a}^T \underline{\underline{K}}\,\underline{a}}{\underline{a}^T \underline{\underline{M}}\,\underline{a}}}$

Case (a): $\underline{K} = \left.\frac{\partial^2 V}{\partial q^2}\right|_{q_0}$ is positive definite

$\Rightarrow \lambda^2 < 0 \Rightarrow \lambda = \pm i\omega \Rightarrow$ $\tilde{x}_1(t) = \underline{a}e^{i\omega t}$
$\tilde{x}_2(t) = \underline{a}e^{-i\omega t}$

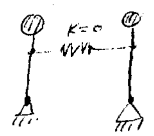
$\tilde{\underline{x}}(t)$: normal mode
$\underline{a}$: mode shape
ω: natural frequency

Case (b) $\underline{\underline{K}} = \frac{\partial^2 V}{\partial q^2}\Big|_{q_0}$ is negative definite $\Rightarrow$ maximum for V at q_0

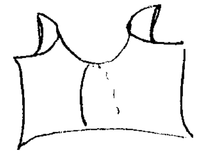
$\Rightarrow \lambda^2 > 0 \Rightarrow \lambda = \pm\nu \Rightarrow \tilde{x}_1(t) = \underline{a}e^{\nu t}$
$\tilde{x}_2(t) = \underline{a}e^{-\nu t}$

$\Rightarrow \underline{x} = \underline{0}$ is unstable

example:



Case (c) $\underline{\underline{K}} = \frac{\partial^2 V}{\partial q^2}\Big|_{q_0}$ is indefinite $\rightarrow V$ has a saddle type nature
(has a point of inflection)



Instability

Example: Coupled pendulum & spring $K \neq 0$

How do we find λ & $\underline{a}$?

Consider the case of pos. def. $\underline{\underline{K}} \Rightarrow (-\omega^2\underline{\underline{M}} + \underline{\underline{K}})\underline{a} = \underline{0}$

$\Rightarrow -\omega^2\underline{\underline{M}} + \underline{\underline{K}}$ is a

Singular matrix $\Rightarrow$
$$\begin{vmatrix} -\omega^2 m_{11} + K_{11} & \cdots & -\omega^2 m_{1n} + K_{1n} \\ \vdots & & \\ -\omega^2 m_{n1} + K_{n1} & \cdots & -\omega^2 m_{nn} + K_{nn} \end{vmatrix} = 0$$

- n^{th} order polynomial for ω^2 (characteristic equation)

$a_1(\omega^2)^n + a_2(\omega^2)^{n-1} \cdots + a_{n+1} = 0$

always has n-roots (all roots are real here - since $\underline{\underline{M}}, \underline{\underline{K}}$ are pos. def.)

$\Rightarrow \omega_1^2 \ldots \omega_n^2$

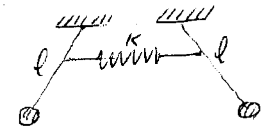
$\Rightarrow \omega_1, \ldots, \omega_n$

n-natural frequencies

Finding the mode shapes: $(-\omega^2\underline{\underline{M}} + \underline{\underline{K}})\underline{a} = \underline{0} \Rightarrow a_j = \text{---}$

terminology $\Phi = [\underline{a}_1, \ldots, \underline{a}_n]$ modal matrix (is not unique, but the directions are unique.)

Example: Reconsider pendulum-spring system



$$\underline{\underline{M}} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix} \qquad K = \begin{pmatrix} mgl + kh^2 & -kh^2 \\ -kh^2 & mgl + kh^2 \end{pmatrix}$$

Let: $ml^2 = 1$ $[kg\,m^2]$ $\Rightarrow$ Char. eq. $\begin{pmatrix} -\omega^2+3 & -1 \\ -1 & -\omega^2+3 \end{pmatrix} = \omega^4 - 6\omega^2 + 8 = 0$

$mgl = 2$ $[N\,m]$

$kh^2 = 1$ $[N\,m]$

$\Rightarrow \omega_1 = \sqrt{2}$ $[1/s]$

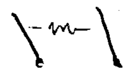
$\omega_2 = 2$ $[1/s]$

Mode shapes: (1) $\omega_1^2 = 2$ $\qquad \begin{pmatrix} -2+3 & -1 \\ -1 & -2+3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underline{0}$

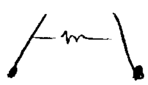
$\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(2) $\omega_2^2 = 4$ $\qquad \begin{pmatrix} -4+3 & -1 \\ -1 & -4+3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underline{0} \longrightarrow \vec{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

a_1



a_2



Eq. of Small Oscillation

$$\underline{M}\ddot{\underline{x}} + \underline{K}\underline{x} = 0$$

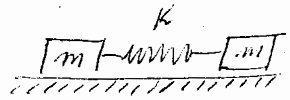
$$\underline{x} = q - q_0$$

$$\tilde{\underline{x}}(t) = \underline{a}e^{\lambda t} = \underline{a}e^{\pm i\omega t}$$

K is positive def.

$$\underline{x}(t) = \sum_j c_j^{\pm} \underline{a}_j e^{\pm i\omega t}$$

Example 2



Guess mode shapes $a_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\omega_1^2 = \frac{2K}{m}$

$\underline{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\omega_2^2 = 0$ rigid body mode

For $\omega_2^2 \neq 0$ $\Rightarrow$ Normal Mode: $\tilde{X}(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\omega_2 t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin\omega_2 t$

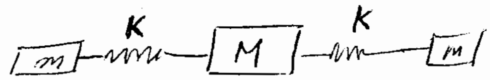
Initial displacement $\tilde{X}(0) = \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_1 \end{pmatrix} \Rightarrow C_1 = x_0$

$$\dot{\tilde{X}}(0) = \begin{pmatrix} V_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} \omega_2 C_2 \\ \omega_2 C_2 \end{pmatrix} \Rightarrow C_2 = \frac{V_0}{\omega_2}$$

$$\Rightarrow \tilde{x}(t) = x_0\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\omega_0 t + \frac{V_0}{\omega_2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin\omega_2 t$$

$$\xrightarrow{\omega_2 \to 0} \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} + \begin{pmatrix} V_0 t \\ V_0 t \end{pmatrix}$$

Example 3.



guess mode shapes & natural frequencies

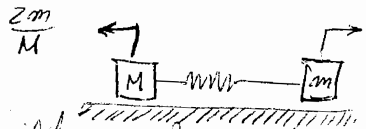
(1) $\underline{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$; $\omega_1^2 = \frac{K}{m}$

(2) Rigid Body mode $\underline{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\omega_2^2 = 0$

(3) $\underline{a}_3 = \begin{pmatrix} 1 \\ -A \\ 1 \end{pmatrix}$

By Conservation of linear momentum if we subtract the rigid body motion from the full motion the CM should not move

$$m \cdot 1 + m \cdot 1 - MA = 0 \Rightarrow A = \frac{2m}{M} \quad \Rightarrow \underline{a}_3 = \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$



the equivalent stiffness for 3rd mass

$$\tilde{K} = K\cdot 1 + k\cdot\frac{2m}{M} = K\left(1+\frac{2m}{M}\right)$$

$\Rightarrow$ For the 1DOF oscillator $m\ddot{x}_3 + \tilde{K}x_3 = 0 \Rightarrow \omega_3^2 = \frac{\tilde{K}}{m} = \frac{K\left(1+\frac{2m}{M}\right)}{m}$

Back to $\quad M\ddot{x} + K\bar{x} = 0 \quad (*)$

$\underline{K}$ pos. definite

$\Downarrow$

- $\underline{a}_1, \dots, \underline{a}_n$ mode shapes
- $\omega_1^2, \dots, \omega_n^2$ natural frequencies (squared)

General Solution (1) $\underline{X}(t) = \sum_{j=1}^{n}\left(p_j a_j e^{i\omega_j t} + q_j a_j e^{-i\omega_j t}\right)$

Complex Constants determined by I.C.

(2) $\Rightarrow p_j = \bar{q}_j = \gamma_j + i\delta_j$

($\underline{X}(t)$ must be real)

Substitution of (2) into (1) gives

$$\underline{X}(t) = \sum_j \left(2\gamma_j \cos\omega_j t - 2\delta_j \sin\omega_j t\right)\underline{a}_j$$

$$= \sum_{j=1}^{n} \underbrace{2\sqrt{\gamma_j^2+\delta_j^2}}_{c_j} \overbrace{\left(\underbrace{\frac{\gamma_j}{\sqrt{\gamma_j^2+\delta_j^2}}}_{\sin(\beta_j)} \cos\omega_j t - \underbrace{\frac{\delta_j}{\sqrt{\gamma_j^2+\delta_j^2}}}_{\cos(\beta_j)} \sin\omega_j t\right)}^{\sin(\omega_j t+\beta_j)} a_j$$

$$\Rightarrow X(t) = \sum_{j=1}^{n} c_j \underline{a}_j \sin(\omega_j t + \beta_j)$$

c_j, β_j real Constants determined by I.C.

Orthogonality of mode shapes

(3) $\quad -\omega_j^2 \underline{M}\,\underline{a}_j + \underline{K}a_j = \underline{0}$

(4) $\quad -\omega_k^2 \underline{M}a_k + \underline{K}a_k = \underline{0}$

$\omega_j \neq \omega_k$

$a_K^T\,(3) - a_j^T\,(4)$:

$$(\omega_2^2 - \omega_1^2)\, a_K^T M a_j = 0$$

Used: $a_K^T \underline{K}\, a_j = a_j^T \underline{K}\, a_K$

$$a_K^T \underline{M}\, a_j = a_j^T \underline{M}\, a_K$$

$\Rightarrow \boxed{a_K^T \underline{M}\, a_j = 0}$ for any $j \neq K$

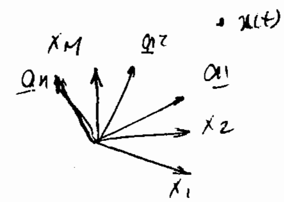
Now: $a_K^T(3) + a_j^T(4)$:

$\boxed{a_K^T \underline{K}\, a_j = 0}$ for $j \neq K$

We can use orthogonality properties to decouple the lin. eq of motion into a system of uncoupled linear oscillations

let $\boxed{x = \Phi y}$

y modal or Principal Coordinates
projections of x onto $a_1, \dots, a_n$



$M\ddot{x} + Kx = 0$.

$$\Rightarrow \underline{M}\,\Phi\,\ddot{y} + \underline{K}\,\Phi\, y = 0$$

left multiply by Φ^T $\quad \Phi^T \underline{M}\, \Phi\, \ddot{y} + \Phi^T \underline{K}\, \Phi\, y = 0$

Note: $\Phi^T M \Phi = \begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_n^T - \end{bmatrix} \begin{bmatrix} & \underline{M} & \end{bmatrix} \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} m_1 & & 0 \\ & m_2 & \\ 0 & & \ddots \; m_n \end{bmatrix}$

Same for $\underline{K}$

$\Rightarrow$ (5) takes the form $\begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_1 \end{pmatrix} \ddot{y} + \begin{pmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_n \end{pmatrix} y = 0$

$$\boxed{\ddot{y}_j + \underbrace{\frac{k_j}{m_j}}_{\omega_j^2} y_j = 0} \quad j = 1, \dots, n$$

$$\underline{M}\ddot{x} + (\underline{G}+\underline{C})\dot{x} + \underline{K}x = \underline{0}$$

$$x = q = \dot{q} =$$

Damped Oscillations (small) (about $q = q_0$; $x = q - q_0$)

(Holonomic scleronomic System)

(1) $\underline{M}\ddot{x} + \underline{C}\dot{x} + \underline{K}x = \underline{0}$ $\qquad$ $\underline{M} = \underline{M}^T$ pos. def.

$\underline{K} = \underline{K}^T \to$ pos. def. $\to$ because System is Conservative

"damping nature"

If I multiply from the left by $\dot{x}^T$

$$\Rightarrow \frac{d}{dt}\Big[\underbrace{\tfrac{1}{2}\dot{x}^T\underline{M}\dot{x} + \tfrac{1}{2}x^T\underline{K}x}_{\text{quadratic terms in total energy}}\Big] = -\dot{x}^T\underline{C}\dot{x} \Rightarrow \underline{C} \text{ is positive semi definite}$$

$$\underline{C} = \underbrace{\tfrac{1}{2}(\underline{C}+\underline{C}^T)}_{\text{sym.}} + \underbrace{\tfrac{1}{2}(\underline{C}-\underline{C}^T)}_{\text{skew-sym.}}$$

$$= -\dot{x}^T\left(\tfrac{1}{2}(\underline{C}+\underline{C}^T)\right)\dot{x}$$

only the symmetric part contribute to energy dissipation

if $\underline{C}$ is positive definite then energy is decreased as long as the velocity is non-zero

Solution of (1) are of the form $\quad x(t) = a e^{\lambda t}$, $\lambda \in \mathbb{C}$

$$= a^{(\operatorname{Re}\lambda + i \operatorname{Im}\lambda)t} = a^{(\operatorname{Re}\lambda)t}\left[\cos \operatorname{Im}\lambda t + i \sin \operatorname{Im}\lambda t\right]$$

$\Rightarrow$ Substitution into (1):

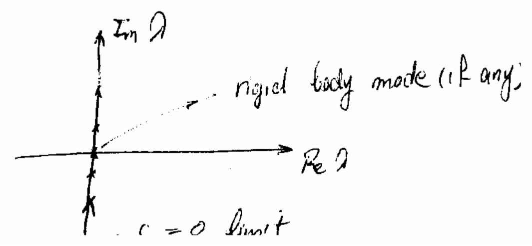
$$(\lambda^2\underline{M} + \lambda\underline{C} + \underline{K})\,\underline{a} = \underline{0} \qquad (\underline{a} \neq \underline{0})$$

$\to \det(\lambda^2\underline{M} + \lambda\underline{C} + \underline{K}) = 0$ $\quad$ Characteristic equation for λ

with $2n$ roots (real or Complex Conjugate pairs)

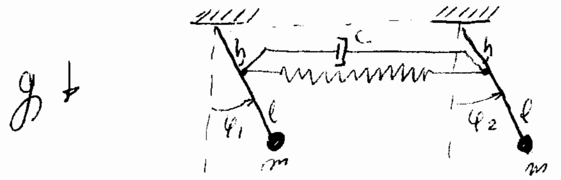
Roots on the Complex plane

ω_*: undamped natural frequency (for $\underline{C} = \underline{0}$)



$\operatorname{Re}\lambda_i < 0$

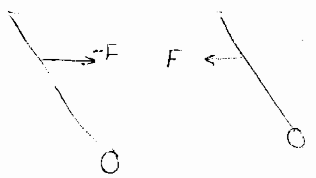
Example: Damped Pendulum-Spring System



c: damping coeff. for dashpot

we have seen $\underline{\underline{M}} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}$; $\underline{\underline{K}} = \begin{pmatrix} mgl+kh^2 & -kh^2 \\ -kh^2 & mgl+kh^2 \end{pmatrix}$

To find $\underline{\underline{C}}$, first identify the generalized (non-potential) force in this system.
active



$$\underline{F} = -c\frac{d}{dt}\left[h(\sin\varphi_2 - \sin\varphi_1)\underline{i} - h(\cos\varphi_2 - \cos\varphi_1)\underline{j}\right]$$

$$= -ch\left[(\cos\varphi_2\dot{\varphi}_2 - \cos\varphi_1\dot{\varphi}_1)\underline{i} - (\sin\varphi_2\dot{\varphi}_2 + \sin\varphi_1\dot{\varphi}_1)\underline{j}\right]$$

$\delta W_I = -\underline{F}\cdot\delta\underline{r}_1 + \underline{F}\cdot\delta\underline{r}_2$ $\quad$ $\delta r_1 = \delta(h\sin\varphi\,\underline{i} - h\cos\varphi\,\underline{j})$

$$= h\,\delta\varphi_1(\cos\varphi_1\underline{i} + \sin\varphi_1\underline{j})$$

$$\delta r_2 = h\,\delta\varphi_2(\cos\varphi_2\underline{i} + \sin\varphi_2\underline{j})$$

$$\delta W_E = ch^2\delta\varphi_1\left[\cos(\varphi_2-\varphi_1)\dot{\varphi}_2 - \dot{\varphi}_1\right] - ch^2\delta\varphi_2\left[\dot{\varphi}_2 - \cos(\varphi_2-\varphi_1)\dot{\varphi}_1\right]$$

$\varphi_1 = x_1 \quad \varphi_2 = x_2$

$$\delta W_E = \delta x_1\underbrace{\left[-ch^2(\dot{x}_1 - \dot{x}_2) + O(2)\right]}_{Q_{x_1}} + \delta x_2\underbrace{\left[-ch^2(\dot{x}_2 - \dot{x}_1) + O(2)\right]}_{Q_{x_2}}$$

For linearized eq. of motion we need $\underbrace{\left.\frac{\partial Q}{\partial \dot{\varphi}}\right|_{\substack{\varphi=\varphi_0=0 \\ \dot{\varphi}=0}}}_{\left.\frac{\partial Q}{\partial \dot{x}}\right|_{\substack{x=0 \\ \dot{x}=0}}} = \underbrace{\begin{pmatrix} -ch^2 & ch^2 \\ ch^2 & -ch^2 \end{pmatrix}}_{-\underline{\underline{C}}}$

$$\underline{\underline{C}} = \begin{pmatrix} ch^2 & -ch^2 \\ -ch^2 & ch^2 \end{pmatrix}; \text{ eig values } \lambda_1, \lambda_2 \Leftarrow$$

because it is generalized force corresponding with damping
So $\underline{\underline{C}}$ is the negative of this value

Symmetric
Singular $\det\underline{\underline{C}} = 0 \rightarrow \lambda_1\lambda_2 = 0$
positive semi definite
$\quad tr(\underline{\underline{C}}) = 2ch^2 = \lambda_1 + \lambda_2$

$\Rightarrow \lambda_1 = 0,\ \lambda_2 > 0 \longrightarrow \underline{\underline{C}}$ pos. semi-definite $\longrightarrow$ acceptable

(makes sense physically these are small independent oscillations)

Forced Small Oscillations ($\underline{\underline{c}} = \underline{\underline{0}}$ for Simplicity)

$$\underline{\underline{M}}\,\ddot{\underline{x}} + \underline{\underline{k}}\,\underline{x} = \underline{F}(t) = \underline{F}\sin\omega t \quad \text{(Sinusoidal Forcing)}$$

pass to modal (principal) Coordinates

$$\underline{x} = \underline{\underline{\Phi}}\,\underline{y}\,; \quad \underline{\underline{\Phi}} = [\underline{a}_1, \dots, \underline{a}_n]$$

As earlier left multiplying (2) by $\underline{\underline{\Phi}}^T$

$$\underbrace{\underline{\underline{\Phi}}^T \underline{\underline{M}}\,\underline{\underline{\Phi}}}_{\begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix}}\ddot{\underline{y}} + \underbrace{\underline{\underline{\Phi}}^T \underline{\underline{k}}\,\underline{\underline{\Phi}}}_{\begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{pmatrix}}\underline{y} = \underbrace{\underline{\underline{\Phi}}^T \underline{F}}_{\underline{f} = \begin{Bmatrix} f_1 \\ \vdots \\ f_n \end{Bmatrix}}\sin\omega t$$

$$\Rightarrow \ddot{y}_j + \frac{k_j}{m_j}y_j = \frac{f_j}{m_j}\sin\omega t\,; \quad j = 1, \dots, n$$

Solution: $y_j(t) = y_j^{hom.}(t) + y_j^{par.}(t)$

$$y_j^{hom}(t) = C_1\cos\omega_j t + C_2\sin\omega_j t\,, \quad \omega_j^2 = \frac{k_j}{m_j}$$

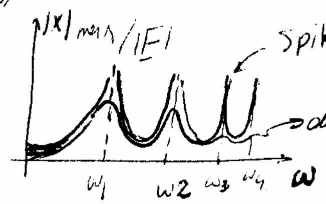
Case (a) $\omega_j \neq \omega \Rightarrow y_j^{par}(t) = A_j\sin\omega t \longrightarrow A_j = \dfrac{f_j/m}{\omega_j^2 - \omega^2}$

$$= \frac{f_j/m}{\omega_j^2 - \omega^2}\sin\omega t$$

Case (b) $\omega_j = \omega$ (resonance) $\Rightarrow y_j^{par.}(t) = (A_j + B_j t)\sin\omega t + (C_j + D_j t)\cos\omega t$

$\Rightarrow \dots$ growing oscillations

Response Diagram



spiky → not dangerous

→ damped response

dangerous