Digital control

29/04/2003

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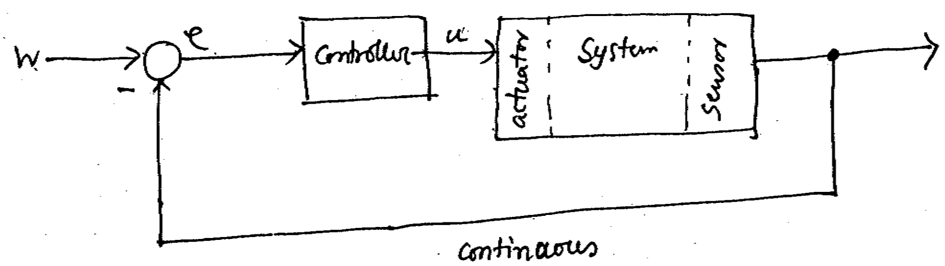
Predictive control

Digital control of Dynamic Systems → Recommended study book

Franklin,
Powell,
Workman
(Addition/Willey)

Control Loop (continuous control)

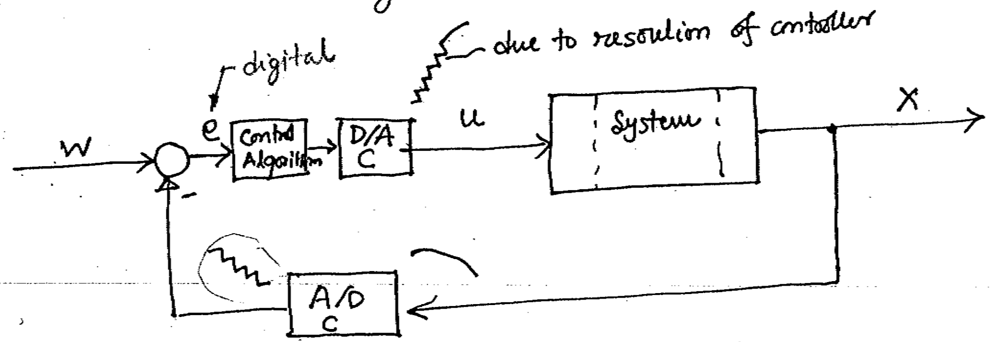


Electronic Realization of the controller!

Advantage: very fast
disadvantage: less changability,
no adaptive control

Relay time – decreases stability

Introduction: Realizing continuous controller digitally changes control loop:



Controllers have limited resolution → stair-function

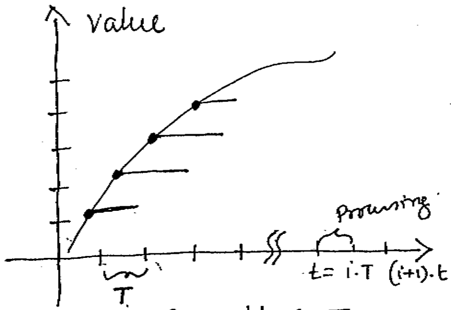
(16 bit is OK)

next class 13 th

Lec-2

13/05/2003

In digital control we use discrete signal.



sampling time T

We assume it is constant.

If it's not, problem – in stabilization

We use interrupter – timer.

Data acquisition, process ~~control~~, of data related to control (?)

$T >$ time for data processing related to control

discretization in value: depending on the signal correction (D/A or A/D) specified in bit resolution

10 bit $\longrightarrow 2^{10}$ steps for the whole range

resolution : $\dfrac{\text{whole range}}{2^{10}}$

2. Quasi-continuous control:

Control Algorithm are realized on a digital controller, but the design procedure as well as the algorithms are the same as continuous control!

$\Rightarrow$ Quasi continuous control.

For example,
Realization of a PID controller on a digital controller:

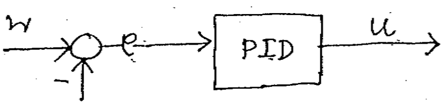
PID: $\underline{U(s)} = G_R(s)\, E(s)$

$$G_R(s) = K_P + \frac{K_I}{s} + K_D s$$

$\frac{1}{s} \rightarrow$

$s \rightarrow$ diff

$\dfrac{U(s)}{E(s)}$



$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)\,d\tau + K_d \frac{de(t)}{dt} \Rightarrow U(s) = K_P E(s) + K_I \frac{E(s)}{s} + K_D s E(s)$$

Sample : $t = i.T$

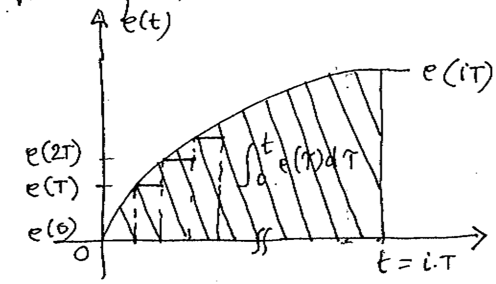
$e(t) = e(i.T) = e_i$ similarly $u(t) = u(iT) = u_i$

(because it started from 0)

$$u_i = K_p \cdot e_i + \overset{K_i}{T} \sum_{j=0}^{i-1} e(jT) + \frac{K_d}{T}\left(e(iT) - e((i-1)T)\right) \rightarrow \text{PID algorithm}$$

(This is now a program)

for integral part :



$$\int_0^t e(\tau)d\tau \simeq T\cdot\left(e(0) + e(T) + e(2T) + e((i-1)T)\right)$$

$$= T\sum_{j=0}^{i-1} e(jT)$$

differential part; $\frac{de(t)}{dt} = \lim_{\Delta t \to 0} \frac{e(t) - e(t-\Delta t)}{\Delta t}$

$\Delta t \rightarrow T$ (but not zero)

$$\simeq \frac{e(iT) - e((i-1)T)}{T}$$

We have a procedure to find K_P, K_I, K_d

Recursive PID Algorithm :

$(i-1)T$: $$u_{i-1} = K_p e_{i-1} + K_I T \sum_{j=0}^{i-2} e_j + \frac{K_D}{T}(e_{i-1} - e_{i-2})$$

$$\Delta u_i = u_i - u_{i-1}$$

$$= K_p(e_i - e_{i-1}) + K_i T \quad e_{i-1} + \frac{K_D}{T}(e_i - 2e_{i-1} + e_{i-2})$$

$$u_i = u_{i-1} + \Delta u_i = u_{i-1} + \underbrace{\left(K_p + \frac{K_D}{T}\right)}_{\text{I can calculate in advance}} e_i + \left(K_I T - K_p - \frac{2K_D}{T}\right) e_{i-1} + \frac{K_D}{T} e_{i-2}$$

$e_i - e_{i-1} - e_{i-1} + e_{i-2}$
$= e_i - 2e_{i-1} + e_{i-2}$

(**) In Exam, Programming of this algorithm

Two other's approximation : (for integral part)

(1) statarting from $e(1T) \cdots\ e(iT)$

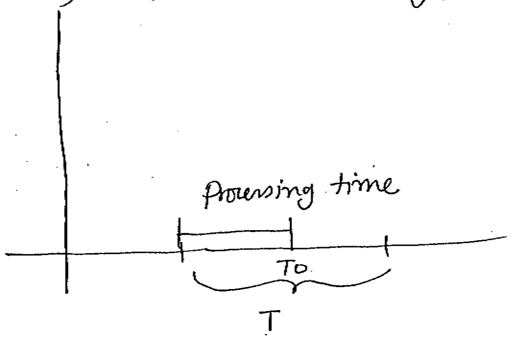
(2) Trapezoidal rule

How to calculate K_P, K_I & K_D:

Design Procedure:

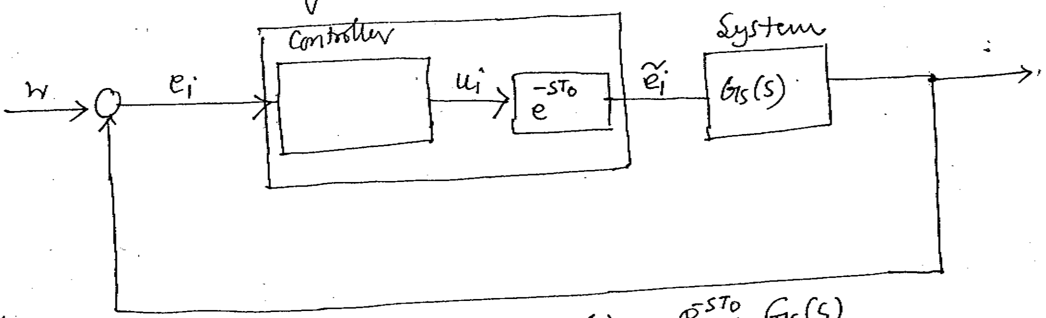
For designing the controller parameters:
we have two alternatives

i) Consider the digital controller as an analog controller, if real processing time is short compared to sampling time



$T_0 \ll T$
(otherwise interrupt will crush the system)

ii) Consider the processing time T_0 as a delay time in the control loop.



System to be controlled: $G_{SN}(s) = e^{-sT_0} \cdot G_S(s)$
(new)

For Quasi continuous controller

- Sampling time should be $\left(\frac{1}{5} \to \frac{1}{10}\right)$ of the fastest time constant, then & only then it makes a sense.

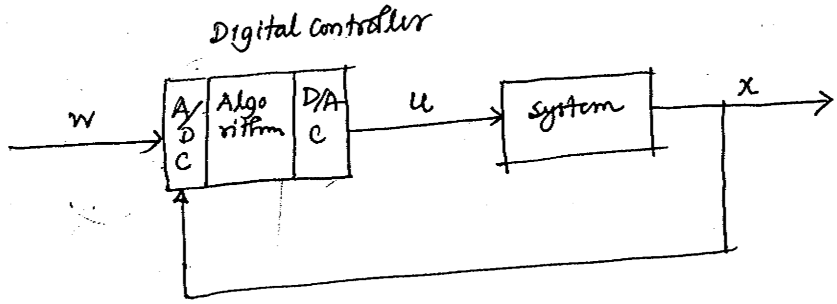
$T \simeq \frac{1}{5} \cdots \frac{1}{10}\ T_{small}$

$T_{small} \to$ the fastest time constant of the system!

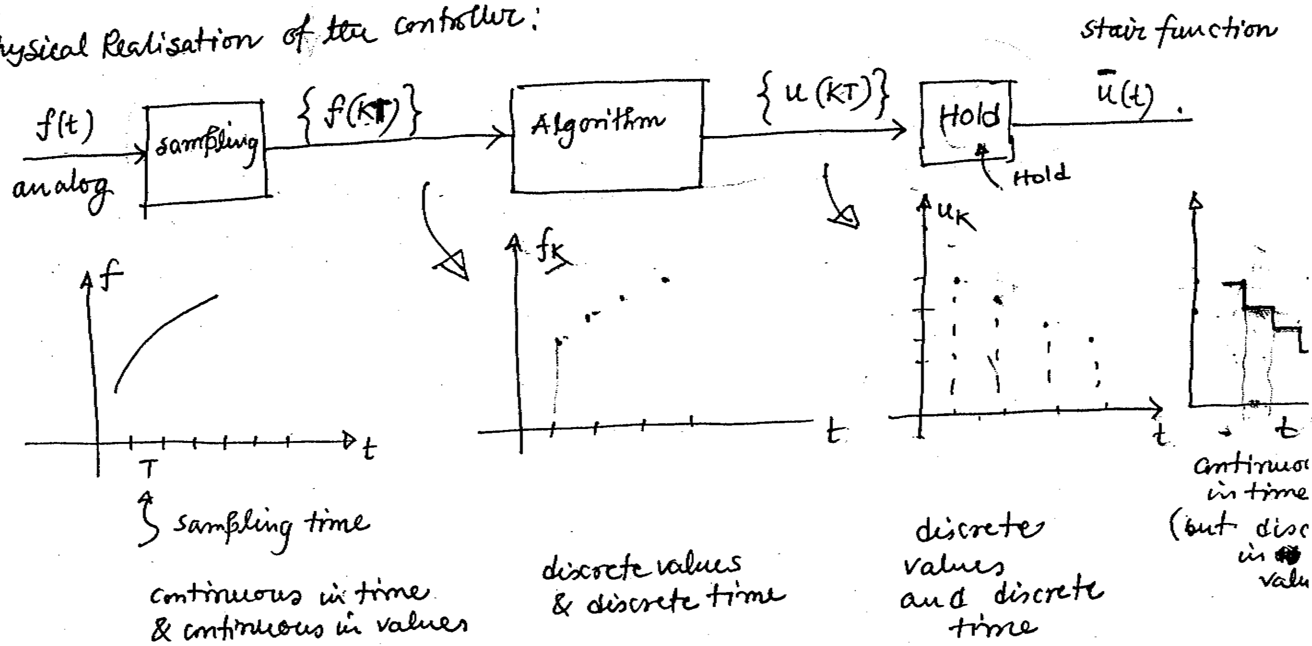
Theory ... when it does not meet the above condition..
next class

100 µs

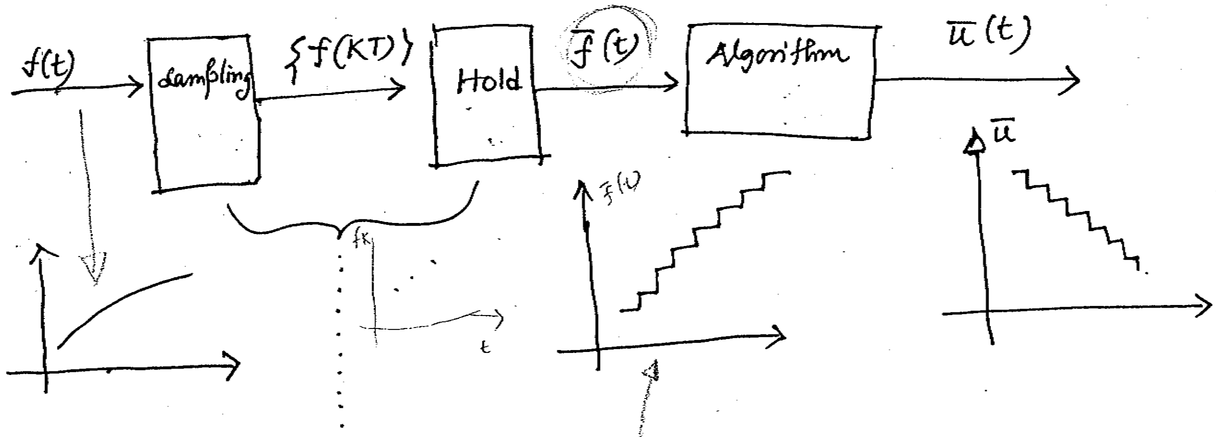
3. Mathematical description of digital control loops:



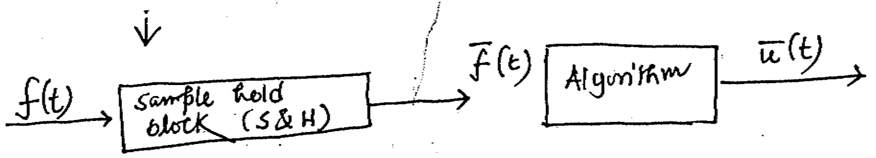
physical Realisation of the controller:



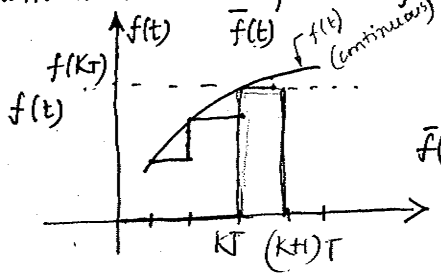
Mathematical description:



The input/output behavior ($f(t) \longrightarrow \bar{u}(t)$)
is the same in both cases!

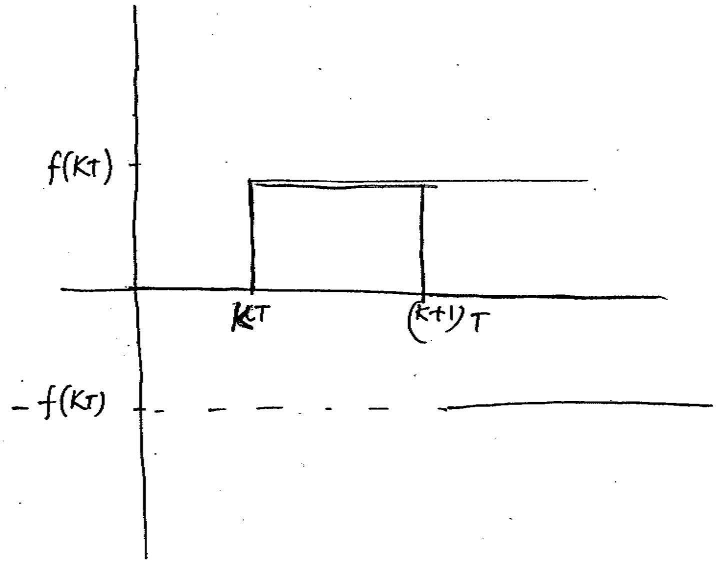


Mathematical description of the S & H - block



step function

$$\bar{f}(t) = \sum_{K=0}^{\infty} f(KT)\left[\delta(t-KT) - \delta(t-(K+1)T)\right]$$



$$\bar{F}(s) = \sum_{K=0}^{\infty} f(KT)\left[\frac{e^{-KTs}}{s} - \frac{e^{-(K+1)Ts}}{s}\right] = \left[\sum_{K=0}^{\infty} f(KT)\, e^{-KTs}\right]\left[\frac{1-e^{-Ts}}{s}\right]$$

$$\boxed{\bar{F}(s) = \frac{1-e^{-Ts}}{s} \underbrace{\sum_{K=0}^{\infty} f(KT)\, e^{-KTs}}_{F^*(s)}}$$

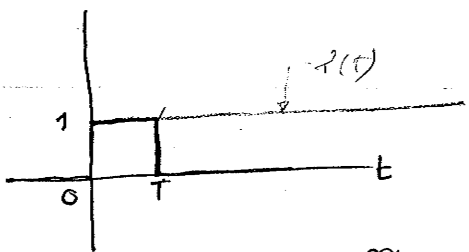
$$\bar{F}(s) = G(s) \cdot F^*(s)$$

$G_H(s)$: Impulse reaction : $G_H(s) = \frac{1}{s} - \frac{e^{-Ts}}{s}$

(Hold block) (Hold)

Impulse in Laplace → 1

$$g_H(t) = \delta(t) - \delta(t-T)$$



$$F^*(s) : \xrightarrow{\mathcal{L}^{-1}} f^*(t) = \sum_{K=0}^{\infty} f(KT)\,\delta(t-KT)$$

f_K

(because of linearity of Laplace)

δ - sampler (not exist in reality)

Algorithms are described as difference equations: (Not differential Equation)
(see PID Algorithm)

$$\Delta u_K = u_K - u_{K-1} = u(Kt) - u((K-1)T)$$

$$\Delta U(s) = U(s) - U(s)e^{-TS}$$

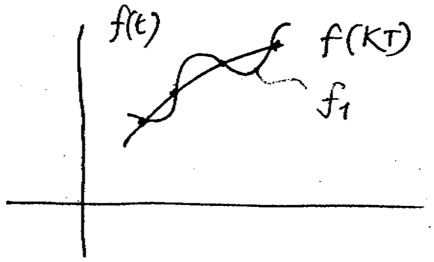
3.1 z Transformation:

Definition $z = e^{TS}$, where T = sampling time (const)

Substitution;

$$F^*(s) \xrightarrow{z=e^{TS}} F_z(z) = \sum_{K=0}^{\infty} f(KT)\, z^{-K}$$

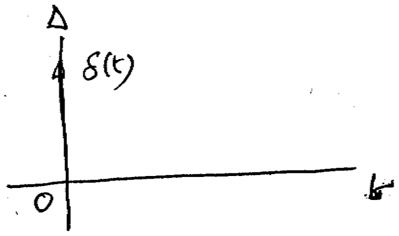
$F_z(z)$ is called the z-Transformation of $F^*(s)$
↳ or ... ... $f^*(t)$
.. .. $f(t)$

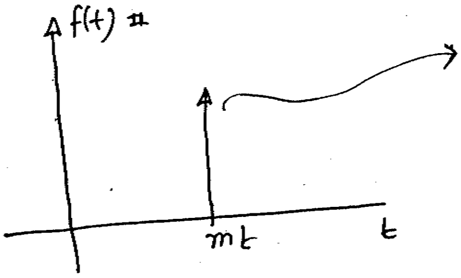


Another Definition: (of z transformation)
take $f(t)$ at the sampling time and multiply it with z^{-K}

Example 1:

$f(t) = \delta(t)$

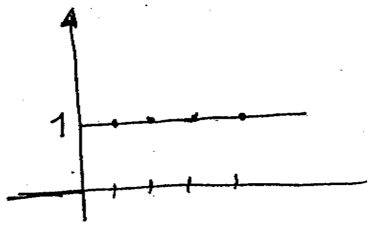




$f(t) = \delta(t - mT)$

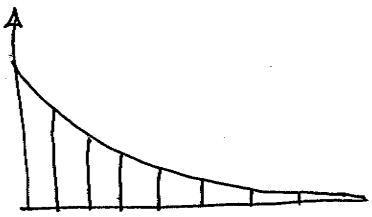
$$\mathcal{Z}\{\delta(t)\} = \mathcal{Z}\{F(s)\} = \mathcal{Z}\{1\} = \mathcal{Z}\{1 \cdot e^{+0TS}\}$$
$$= z^{0} \qquad [z = e^{TS}]$$
$$= 1.$$

Ex. $f(t) = u(t)$



$$\mathcal{Z}\{u(t)\} = \sum_{K=0}^{\infty} 1^K z^{-K} = \sum_{K=0}^{\infty} z^{-K} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^\infty}$$
$$(z^{-1})^K \qquad = \frac{z}{z-1}.$$

Ex: 3: $f(t) = e^{\alpha t} = e^{\alpha KT}$



$$z\{f(t)\} = \sum_{K=0}^{\infty} e^{\alpha T K} z^{-K}$$
$$= \sum_{K=0}^{\infty} (\underbrace{e^{\alpha T} z^{-1}}_{q})^K$$
$$= \sum_{K=0}^{\infty} q^K = \frac{1}{1-q} \qquad [|q| < 1 \text{ then converges}]$$
$$= \frac{1}{1 - e^{\alpha T} z^{-1}} = \frac{z}{z - e^{\alpha T}}$$

Rule for z-transformation:

1) Linearity: Superposition/Amplification

$$\mathcal{Z}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1 \mathcal{Z}\{f_1(t)\} + c_2 \mathcal{Z}\{f_2\} + \cdots + c_n \mathcal{Z}\{f_n(t)\}$$

Proof:

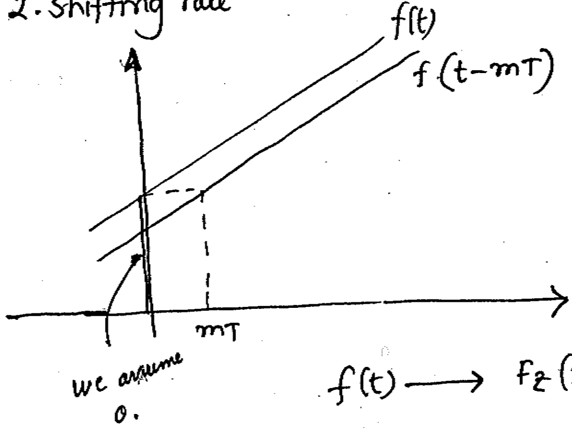
$$\mathcal{Z}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = \sum_{K=0}^{\infty} \left[c_1 f(KT) + \cdots + c_n f_n(KT)\right] z^{-K}$$

$$= \sum_{K=0}^{\infty} c_1 f_1(KT) z^{-K} + \cdots + \sum_{K=0}^{\infty} c_n f_n(KT) z^{-K}$$

$$= c_1 \mathcal{Z}\{f_1(t)\} + \cdots + c_n \mathcal{Z}\{f_n(t)\}$$

(proved)

2. Shifting rule



(Shift to the right)

$$f(t) \longrightarrow F_z(z) = \mathcal{Z}\{f(t)\}$$

$$f(t-mT) \longrightarrow \mathcal{Z}\{f(t-mT)\}$$

$$f(t) \longrightarrow \{f_K\} = \{f_0, f_1, \ldots f_m \ldots \}$$

$$f(t-mT) \rightarrow \{$$

$$\downarrow$$

$$f(KT-mT)$$

$$f((K-m)T) \rightarrow \{f_{-m}, \ldots, f_0, f_1 \ldots$$

$$\downarrow$$

$$f_{-m} = f(-mT)$$

if $f(t) = 0$ for $t < 0$; $f_{-m} = 0$

$f_{-1} = 0$

Let first assume $f(t) \neq 0$ for $t < 0$

$$\mathcal{Z}\{f(t-mT)\} = \sum_{K=0}^{\infty} f(KT - mT) z^{-K}$$

$$= \sum_{K=0}^{\infty} f_{K-m} z^{-K}$$

Assume,

$i = k - m$

cont. $= \sum_{i=-m}^{\infty} f_i \, z^{-(i+m)}$

$= z^{-m} \sum_{i=-m}^{\infty} f_i \, z^{-i}$

$= z^{-m} \sum_{i=0}^{\infty} f_i \, z^{-i} + z^{-m} \sum_{i=-m}^{-1} f_i \, z^{-i}$

$= \boxed{z^{-m} \mathcal{Z}\{f(t)\} + z^{-m} \sum_{i=-m}^{-1} f_i \, z^{-i}}$

In case, $f(t) = 0$, $t < 0$

2nd term cancels out.

Shift to the left:

$$\mathcal{Z}\{f(t+mT)\} = z^{m}\left[F_Z(z) - \sum_{i=0}^{m-1} f_i \, z^{-i}\right]$$

3. Damping rule:

$$f(t) \Rightarrow \mathcal{Z}\{f(t)\} = F_Z(z)$$

$$f(t)e^{\alpha t} \rightarrow \mathcal{Z}\{f(t)e^{\alpha t}\} = F_Z(z e^{-\alpha T})$$

Example: $f(t) = \delta(t) \longrightarrow F_Z(z) = \dfrac{z}{z-1}$ ✓

$$\delta(t)e^{\alpha t} \rightarrow \mathcal{Z}\{\delta(t)e^{\alpha t}\} = \frac{z e^{-\alpha T}}{z e^{-\alpha T} - 1} = \frac{z}{z - e^{\alpha T}}$$

Another Example(5) $f(t) = \sin(\omega t + \phi) = \dfrac{1}{2j}\left(e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right)$

$$f(t) = \frac{1}{2j}\left(e^{j\phi}.e^{j\omega t} - e^{-j\phi}.e^{-j\omega t}\right)$$

$$\mathcal{Z}\{f(t)\} = \frac{1}{2j}\left(e^{j\phi}\frac{z}{z - e^{j\omega T}} - e^{-j\phi}\frac{z}{z - e^{-j\omega T}}\right)$$

$$= z\,\frac{z\sin\phi + \sin(\omega t - \phi)}{z^2 - 2z\cos\omega T + 1}$$

$\mathcal{Z}\{e^{j\phi}.e^{j\omega t}\}$

$= e^{j\phi}\,\mathcal{Z}\{e^{j\omega t}\} = e^{j\phi}.\dfrac{z}{z - e^{j\omega t}}$

$\mathcal{Z}\{e^{-j\phi}.e^{-j\omega t}\}$

$= e^{-j\phi}\,\mathcal{Z}\{e^{-j\omega t}\} = e^{-j\phi}.\dfrac{z}{z - e^{-j\omega t}}$

$\dfrac{1}{2j} \times z \left\{ \dfrac{z e^{j\phi} - e^{-j(\omega t - \phi)} - z e^{-j\phi} + e^{+j(\omega t - \phi)}}{z^2 - z e^{+j\omega t} - z e^{-j\omega t} + 1} \right.$

$= \dfrac{z[z\sin\phi + \sin(\omega t - \phi)]}{z^2 - 2z\cos\omega T + 1}$

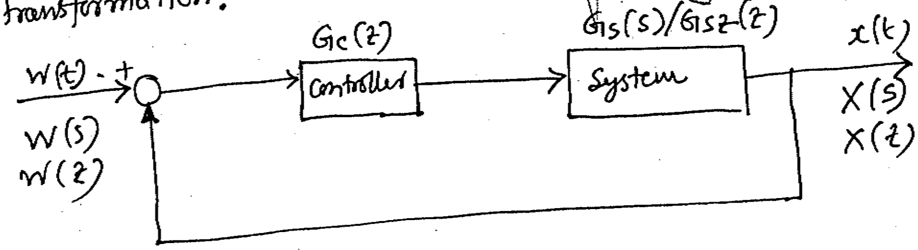
Next class Exercise

Lec-5

03/06/2003

Martin Ruel

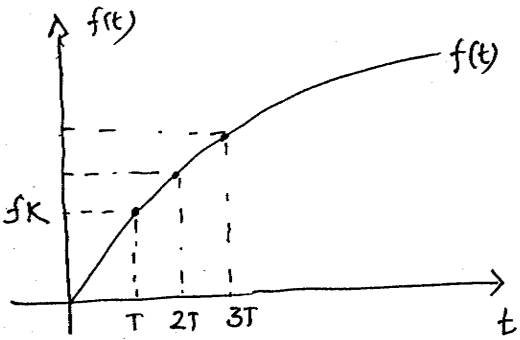
Z-transformation:



Definition of Z-transformation:

$$F(z) = \sum_{k=0}^{\infty} f_k z^{-k}$$

sampled data



Geometric Power Series:

(I) $\displaystyle\sum_{k=0}^{\infty} q^k = 1 + q + q^2 + \cdots + q^k \quad \Big| \; |q| < 1$

(II) $\displaystyle q\sum_{k=0}^{\infty} q^k = q + q^2 + \cdots + q^{k+1}$

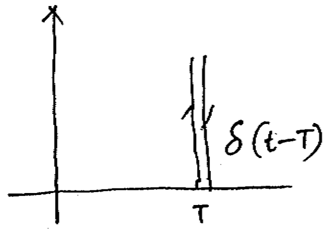
(I) − (II) $\displaystyle\sum_{k=0}^{\infty} q^k - q\sum_{k=0}^{\infty} q^k = 1 - q^{k+1}$

$\displaystyle\Rightarrow \sum_{k=0}^{\infty} q^k (1 - q) = 1 \qquad [\because |q| < 1]$

$\displaystyle\Rightarrow \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \qquad [\because |q| < 1]$

z- transformation of given function $f(t)$ in z domain

first step: develope sampled data function:

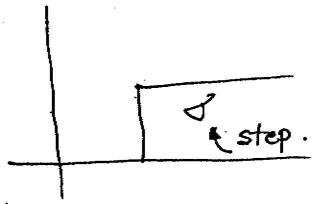


$$f^*(t) = \sum_{K=0}^{\infty} f_K \, \delta(t - KT)$$

impulse function

- second step calculation of power series

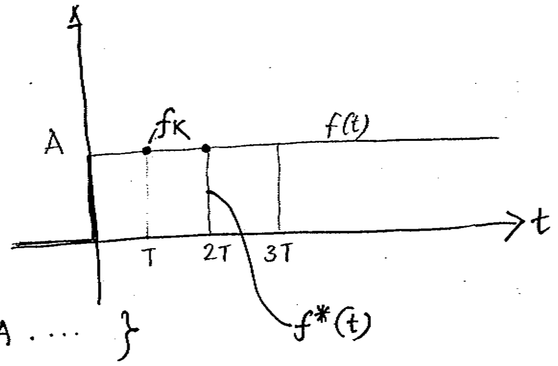
$$\left[\{f_K\} = \{0, 1, 2, 3, \cdots\} \right]$$



1st task:

calculate the z transformation of the following function:

a) $f(t) = A \cdot \sigma(t)$



$$\{f_K\} = \{A, A, A \cdots\}$$

$$f^*(t) = \sum_{K=0}^{\infty} A \cdot \delta(t - KT)$$

$$\Rightarrow F(z) = \sum_{K=0}^{\infty} f_K z^{-K} = A \cdot \sum_{K=0}^{\infty} z^{-K}$$

$$= A \cdot \frac{1}{1 - z^{-1}} = A \cdot \frac{z}{z-1}$$

$q = z^{-1}$; $|q| < 1$; $|z| > 1$

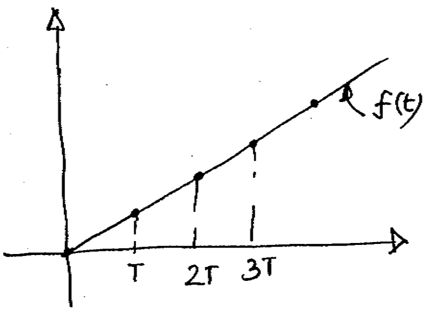
b) $f(t) = e^{\alpha t} \Rightarrow f^*(t) = \sum_{K=0}^{\infty} e^{\alpha KT} \delta(t - KT)$

$$F(z) = \sum_{K=0}^{\infty} f_K z^{-K} = \sum_{K=0}^{\infty} e^{\alpha KT} z^{-K}$$

$$= \sum_{K=0}^{\infty} \underbrace{\left[e^{\alpha T} \cdot z^{-1} \right]}_{q}^{K}$$

$$= \frac{1}{1 - e^{\alpha T} z^{-1}} = \frac{z}{z - e^{\alpha T}}$$

c) $f(t)=t$



$$f^*(t)=\sum_{K=0}^{\infty} KT\,\delta(t-KT)$$

$$\{f_K\}=\{0,T,2T,3T,\cdots\}$$

$$F(z)=\sum_{K=0}^{\infty} f_K z^{-K} = T(z^{-1}+2z^{-2}+3z^{-3}+\cdots)$$

$$=T\underbrace{(1+z^{-1}+z^{-2}+\cdots)}_{q=z^{-1}}\underbrace{(z^{-1}+z^{-2}+z^{-3}+\cdots)}$$

$$=T\cdot\frac{1}{1-z^{-1}}\cdot\frac{z^{-1}}{1-z^{-1}} = T\cdot\frac{z}{z-1}\cdot\frac{1}{z-1}$$

$$=\frac{Tz}{(z-1)^2}$$

d) $f(t)=\cos(\omega t)$

$$f^*(t)=\sum_{K=0}^{\infty}\cos(\omega KT)\,\delta(t-KT)$$

$$\{f_K\}=\{\cos(\omega KT)\}=\left\{\frac{e^{j\omega KT}+e^{-j\omega KT}}{2}\right\}$$

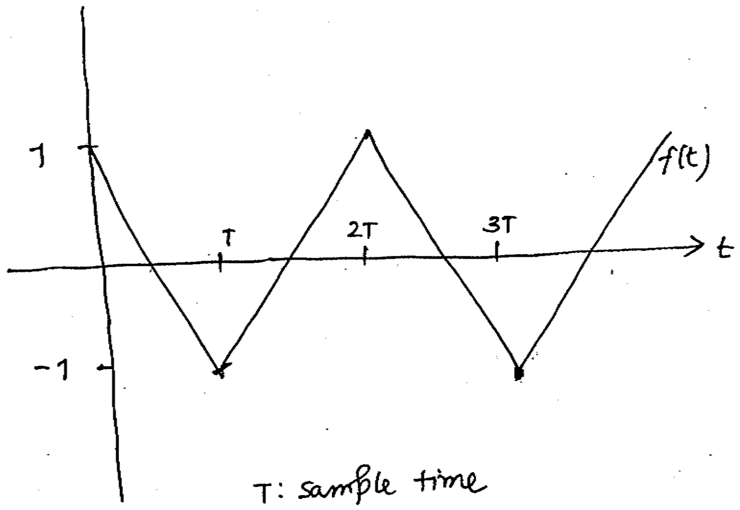
$$F(z)=\frac{1}{2}\Big(\underbrace{\sum_{K=0}^{\infty} e^{j\omega KT} z^{-K}}_{q z^{-K}\ \text{(assume)}}+\sum_{K=0}^{\infty}e^{-j\omega KT}z^{-K}\Big)$$

$$=\frac{1}{2}\left(\frac{z}{z-e^{j\omega T}}+\frac{z}{z-e^{-j\omega T}}\right)$$

$$=\frac{1}{2}\left(\frac{z(z-e^{-j\omega T})+z(z-e^{j\omega T})}{z^2-z(e^{j\omega T}+e^{-j\omega T})+1}\right)$$

$$=\frac{\cancel{2}z(z-\cos(\omega T))}{\cancel{2}(z^2-2z\cos(\omega T)+1)} = \frac{z[z-\cos(\omega T)]}{z^2-2z\cos(\omega T)+1}$$

e)



T: sample time

$$\{f_K\} = \{1, -1, 1, -1 \cdots\}$$

$$F(z) = \sum_{K=0}^{\infty} f_K z^{-K} = 1 - z^{-1} + z^{-2} - z^{-3} + \cdots$$

geometric power series: $q = -z^{-1}$

$$F(z) = \frac{1}{1+z^{-1}} = \frac{z}{z+1} \quad ; \quad |z| > 1$$

2nd Task |

Calculate the continuous function $f(t)$ of a given z-transformation (Inverse z transformation)

a) $F(z) = \dfrac{z}{z^2 - 6z + 5}$ $\neq$

$$z^2 - 6z + 5 = 0$$

$$z_{1/2} = 3 \pm \sqrt{9-5} \qquad z_1 = 5, \quad z_2 = 1$$

1. Partial fraction expansion:

$$F(z) = \frac{z}{(z-5)(z-1)} = \frac{A \cdot z}{z-5} + \frac{B \cdot z}{z-1}$$

$$A = \lim_{z \to 5} \frac{1}{z} \cdot F(z)(z-5) = \frac{1}{z-1} \Big|_{z=5} = \frac{1}{4}$$

$$B = \lim_{z \to 1} \frac{1}{z} F(z)(z-1) = \frac{1}{z-5} \Big|_{z=1} = -\frac{1}{4}$$

$$F(z) = \frac{z}{4(z-5)} - \frac{z}{4(z-1)} \qquad F(z) = \frac{z}{z-c} \multimap \{f_k\} = \{c^k\}$$

$$\Rightarrow \{f_K\} = \left\{\frac{1}{4} 5^k - \frac{1}{4} 1^k\right\} = \frac{1}{4}\{5^k - 1\}$$

Power series:

$$F(z) = \frac{z}{z^2-6z+5} = \frac{z^{-1}}{1-6z^{-1}+5z^{-2}}$$

$$z^{-1} \quad : (1-6z^{-1}+5z^{-2}) = \underbrace{0.z^{0}+}\; 1.z^{-1} + 6z^{-2} + 31z^{-3} + \cdots$$

$$-(z^{-1} - 6z^{-2} + 5z^{-3})$$

$$6z^{-2} - 5z^{-3}$$

$$-(6z^{-2} - 36z^{-3} + 30z^{-4})$$

$$31z^{-3} - 30z^{-4}$$

$$\Rightarrow \{f_k\} = \{0, 1, 6, 31 \cdots\}$$

b) $F(z) = \dfrac{(1-e^{-T})z}{(z-1)(z-e^{-T})}$

1. fraction Expansion:

$$F(z) = z\left(\frac{1}{z-1} - \frac{1}{z-e^{-T}}\right) = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

$$\{f_k\} = \{1^k - e^{-kT}\}$$

$$= \{1 - e^{-kT}\} = \{0, 1-e^{-T}, 1-e^{-2T}, \cdots\}$$

$$\Rightarrow f(t) = 1 - e^{-t}$$

2. Power Series:

$$F(z) = \frac{(1-e^{-T})z^{-1}}{1-(1+e^{-T})z^{-1}+e^{-T}z^{-2}}$$

প্রথমে 1 বানাও, সুবিধা!

(b) 1 এবং 2 same problem

$$(1-e^{-T})z^{-1} \quad : \left[1-(1+e^{-T})z^{-1} + e^{-T}z^{-2}\right]$$

$$(1-e^{-T})z^{-1} - (1-e^{-2T})z^{-2} + e^{-T}(1-e^{-T})z^{-3} \quad = 0.z^{0} + (1-e^{-T})z^{-1} + (1-e^{-2T})z^{-2} + \cdots$$

$$(1-e^{-2T})z^{-2} - e^{-T}(1-e^{-T})z^{-3}$$

$$(1-e^{-2T})z^{-2} + \cdots -$$

$$\{f_k\} = \{0, (1-e^{-T}), (1-e^{-2T}) \cdots)$$

4) Difference Rule:

Differentiation:

$$\dot{f} = \lim_{\Delta t \to 0} \frac{f(t) - f(t-\Delta t)}{\Delta t}$$

$\Delta t = T$ here

$f_k = f(t) = f(kT)$, $t = kT$

$$\dot{f} \simeq \frac{1}{T}\left(f(t) - f(t-T)\right)$$

$$\mathcal{Z}\{f(t)\} = F_z(z) = f_z$$

$$\mathcal{Z}\{f(t-T)\} = z^{-1} F_z(z) + f_{-1}$$

$f(-T) = f_{-1}$

$f(t) = f(kT) = f_k$

$$\therefore \mathcal{Z}\{f(t) - f(t-\Delta t)\} = F_z(z) - \left(z^{-1} F_z(z) + f_{-1}\right)$$

$$= F_z(z)\left(1 - z^{-1}\right) - f_{-1}$$

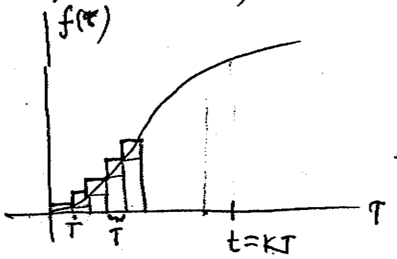
$$= F_z(z) \times \frac{z-1}{z} - f_{-1}$$

$$1 - \frac{1}{z} = \frac{z-1}{z}$$

$$\mathcal{Z}\{f(t) - f(t-\Delta t)\} = f_z(z) \cdot \frac{z-1}{z} - f$$

5) Summation Rule:

(Integral relation)



$$s(t) = \int_0^t f(\tau)\, d\tau \simeq$$

$$\simeq T \sum_{i=0}^{k-1} f_i \simeq T \sum_{i=1}^{k} f_i$$

$$s(0) \cong T \sum_{i=0}^{-1} f_i = 0$$

$$s(1.T) = T \sum_{i=0}^{0} f_i = T f_0$$

$$s(0) = T \sum_{i=1}^{0} f_0 = T f_0$$

$$s(1.T) = s_1 = T(f_0 + f_1)$$

$$\{s_k\} = \{s_0; s_1; s_2 \ldots\}$$

$$\{s_k\} = \{T f_0; T(f_0 + f_1); T(f_0 + f_1 + f_2) \ldots\}$$

$$s_k = T \sum_{i=0}^{k} f_i$$

$$\boxed{\sum_{i=0}^{k} f_i} = \sum_{i=0}^{k} f(iT) = f_0 + f_1 + \cdots + f_k$$

$$\frac{1}{T} s_k = \sum_{i=0}^{\infty} f\left((k-i)T\right) = f_k + f_{k-1} + \cdots + f_0 + \underbrace{f_{-1} + f_{-2} + \cdots}_{0}$$

$$\mathcal{Z}\{s_k\} = \mathcal{Z}\{s(kT)\} = \mathcal{Z}\{s(t)\} = S_z(z)$$

$$\mathcal{Z}\{f_k\} = \mathcal{Z}\{f(kT)\} = \mathcal{Z}\{f(t)\} = F_z(z)$$

$$\mathcal{Z}\{f_{k-1}\} = \mathcal{Z}\{f((k-1)T)\} = \mathcal{Z}\{f(t-T)\} = z^{-1} F_z(z)$$

$$\mathcal{Z}\{f_0\} = \cancel{\mathcal{Z}\{f} \quad z^{-k} F_z(z)$$

$$\frac{1}{T}\mathcal{Z}\{s_k\} = \mathcal{Z}\{f_k + f_{k-1} + \cdots + f_0\}$$

$$\boxed{\frac{1}{T}\mathcal{Z}\{s_k\} = F_z(z)\left(1 + z^{-1} + z^{-2} + \cdots\right) = F_z(z) \times \frac{z}{z-1}}$$

$$\sum_{i=0}^{k} f_i \quad \circ\!\!-\!\!\!-\!\!\!-\!\!\bullet \quad F_z(z)\frac{z}{z-1}$$

Example:

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)\,d\tau + K_D \dot{e}(t)$$

$$u_k = K_P e_k + K_I T \sum_{i=0}^{k-1} e_i + \frac{K_D}{T}\left(e_k - e_{k-1}\right)$$

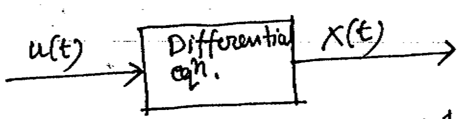
$\downarrow$ $\mathcal{Z}$ transformation

$$U_z(z) = K_P E_z(z) + K_I T\,\mathcal{Z}\left\{\left(\sum_{i=0}^{k} e_i - e_k\right)\right\} + \frac{K_D}{T}\mathcal{Z}\{e_k - e_{k-1}\}.$$

$$= K_P E_z(z) + K_I T\left[E_z(z)\frac{z}{z-1} - E_z(z)\right] + \frac{K_D}{T}\,\frac{z-1}{z}$$

$$U_z(z) = \underbrace{\left[K_P + K_I T\left(\frac{z}{z-1} - 1\right) + \frac{K_D}{T}\,\frac{z-1}{z}\right]}_{\mathcal{Z}\text{ transfer function of a PID Controller.}} E_z(z)$$

6) Convolution Rule:



- $X(s) = G(s)\cdot U(s)$ $\qquad X(s) \xrightarrow{\mathcal{L}^{-1}} x(t)$
- Classical analytical soln
- Convolution:

$x(t) = u(t) \overset{*}{} g(t)$ ← Convolution

$\mathcal{L}\{g(t)\} = G(s)$

Convolution in time $\equiv$ multiplication in Laplace

Fact: Convolution in time domain corresponds to multiplication in Laplace domain.

Question: Does convolution in time domain also corresponds to multiplication in Z domain?

Ans → No. in general! only if at least one of the participating functions are "*" function (star function → dirac Impulse)

Then $Z\{x(t)\} = Z\{g(t) * u(t)\} = Z\{g(t)\} \cdot Z\{u(t)\}$

if $g = g^*$ or $u = u^*$ or both.

(7) Limit Sentences:

Initial value: $\lim_{z \to \infty} F_Z(z) = f_0$ $\quad \lim_{t \to 0} f(t)$

Final value: $\lim_{K \to \infty} f_K = \lim_{z \to 1+} (z-1) * F_Z(z)$

right limit value

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Z-Transformation of transfer function resulting from Difference eqn

From differential equations

$$u(t) \longrightarrow \boxed{\text{Differential eqn}} \longrightarrow x(t)$$

differential eqn :

$$a_0 x + a_1 \dot{x} = b_0 u + b_1 \dot{u} \quad \text{(first order)} \qquad a_1 \dot{x} + a_0 x = b_1 \dot{u} + b_0 u$$

$$\dot{x} = \cancel{\lim_{\Delta t \to 0}} \frac{x(t) - x(t-\Delta t)}{\Delta t} = \frac{x(t) - x(t-T)}{T}$$

$$\Delta t \downarrow T$$

$$x(t) = x(KT) = x_K$$

$$a_0 x(t) + a_1 \frac{x(t) - x(t-T)}{T} = b_0 u(t) + b_1 \frac{u(t) - u(t-T)}{T}$$

$$a_0 x_K + \frac{a_1}{T}(x_K - x_{K-1}) = b_0 u_K + \frac{b_1}{T}(u_K - u_{K-1})$$

$$x_K = \frac{1}{a_0 + \frac{a_1}{T}} \left[\frac{a_1}{T} x_{K-1} + \left(b_0 + \frac{b_1}{T}\right) u_K - \frac{b_1}{T} u_{K-1} \right]$$

It's very very coarse approximation,

Generalization :

$$a_n x^{(n)} + \ldots + a_1 \dot{x} + a_0 x = b_1 u + \ldots + b_m u^{(m)}$$

difference equation
$$\boxed{a_n^* x_{K-n} + a_{n-1}^* x_{K-(n-1)} + \cdots + a_1^* x_{K-1} + a_0^* x_K = b_m^* u_{K-m} + \cdots + b_1^* u_{K-1} + b_0^* u_K}$$

Now, we want to transform it into z domain:

Laplace transformation :

$$a_n^* X(s) e^{-smT} + a_{n-1}^* X(s) e^{-s(n-1)T} + \ldots + a_1^* X(s) e^{-sT} + a_0^* X(s) =$$

$$= b_m^* U(s) e^{-smT} + \ldots + b_1^* U(s) e^{-sT} + b_0^* U(s)$$

$$X(s) = \underbrace{\frac{b_m^* e^{-smT} + \ldots + b_1^* e^{-sT} + b_0^*}{a_m^* e^{-smT} + \ldots + a_1^* e^{-sT} + a_0^*}}_{G(e^{sT})} U(s)$$

Put: $z = e^{sT}$

$$Z\{X(s)\} = \boxed{X_z(z) = G(z) \cdot U_z(z)}$$

This is allowed, if $G(e^{sT})$ or $U(s)$ are star functions

Polynomial division

$$G(e^{sT}) = g_0 + g_1 e^{-sT} + g_2 e^{-s2T} + \cdots$$

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$$\Downarrow \mathcal{L}^{-1}$$

$$g(t) = g_0\delta(t) + g_1\delta(t-T) + g_2\delta(t-2T) + \cdots = g^*(t)$$

Substituting $e^{sT} = z$

$$\boxed{G(z) = \frac{b_m^* z^{-m} + \cdots + b_1^* z^{-1} + b_0^*}{a_m^* z^{-m} + \cdots + a_1^* z^{-1} + a_0^*}}$$

$G(z)$

$g(t)$

z-transformation of a $G_1(s) \cdot G_1(e^{sT})$

$$\mathcal{Z}\{G_1(s) \cdot G_1(e^{sT})\} = \mathcal{Z}\{G_1(s)\} \cdot G_2(z)$$

$$\frac{1}{s} e^{-sT}$$

$$\downarrow$$

$$g_1\delta(t-T)$$

$$\mathcal{Z}\{g_1(t) * g_2^*(t)\} = \mathcal{Z}\{g_1(t)\}\,\mathcal{Z}\{g_2^*(t)\}$$

Example: step response of a S & H (sample & hold)

$$G_{S\&H} = \frac{1-e^{-sT}}{s}$$

$$U(s) = \frac{1}{s}$$

$$X(s) = G_{S\&H}(s) \cdot \frac{1}{s} = \frac{1-e^{-sT}}{s} \cdot \frac{1}{s}$$

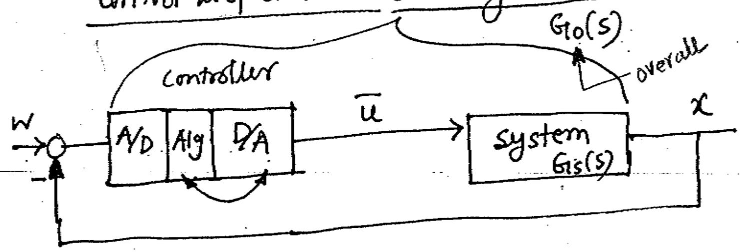
$$X_z(z) = \mathcal{Z}\left\{\frac{1-e^{-sT}}{s} \cdot \frac{1}{s}\right\} = \mathcal{Z}\left\{(1-e^{-sT})\frac{1}{s^2}\right\}$$

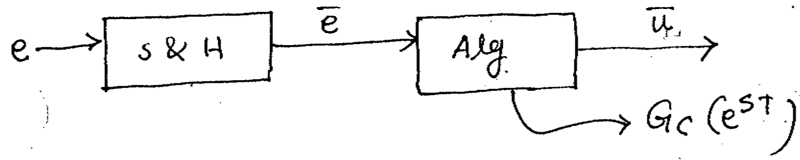
$$X_z(z) = (1-z^{-1}) \cdot \mathcal{Z}\left\{\frac{1}{s^2}\right\} = (1-z^{-1}) \cdot \frac{Tz}{(z-1)^2}$$

$$X_z(z) = \frac{T}{z-1}$$

from Table (Rule 2)

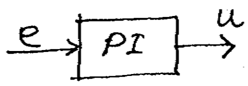
control loop considered in z domain

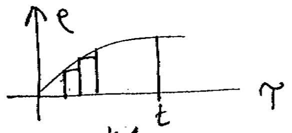




Example: Algorithm shall be a PI controller

$$u(t) = K_p e(t) + K_I \underbrace{\int_0^t e(\tau)\,d\tau}$$





$$u(kT) = K_p e(kT) + K_I T \sum_{i=0}^{k-1} e(iT)$$

$$u_k = K_p e_k + K_I T \sum_{i=0}^{k-1} e_i$$

$k-1$: $u_{k-1} = K_p e_{k-1} + K_I T \sum_{i=0}^{k-2} e_i$

$$u_k - u_{k-1} = \Delta u_k = K_p(e_k - e_{k-1}) + K_I T e_{k-1} \overset{Alg}{\Rightarrow} u_k = u_{k-1} + K_p(e_k - e_{k-1}) + K_I T e_{k-1}$$

Now into z domain:

first into $\mathcal{L}$: $U(s) = U(s)e^{-sT} + K_p\left(E(s) - E(s)e^{-sT}\right) + K_I T E(s) e^{-sT}$

$$G_c = \frac{U(s)}{E(s)} = \frac{K_p - K_p e^{-sT} + K_I T e^{-sT}}{1 - e^{-sT}} = \frac{K_p + (K_I T - K_p)e^{-sT}}{1 - e^{-sT}}$$

$$= \frac{K_p + (K_I T - K_p) z^{-1}}{1 - z^{-1}}$$

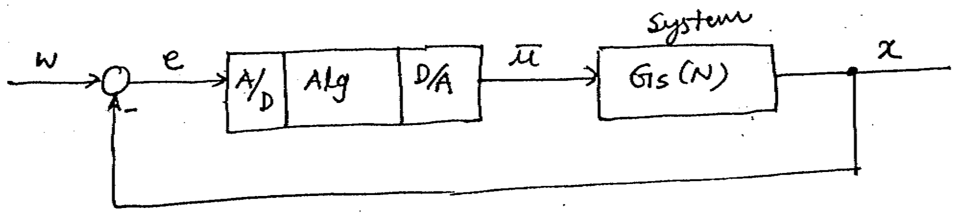
$$G_c = G_c(e^{sT})$$
↓
star function

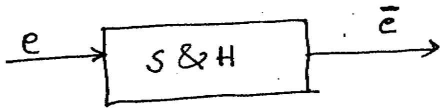
$$G_0(s) = G_{S\&H}(s) \cdot G_c(e^{sT}) \cdot G_s(s) = \left(\frac{1 - e^{-sT}}{s}\right) \cdot G_c(e^{sT}) \cdot \frac{G_s(s)}{s}$$

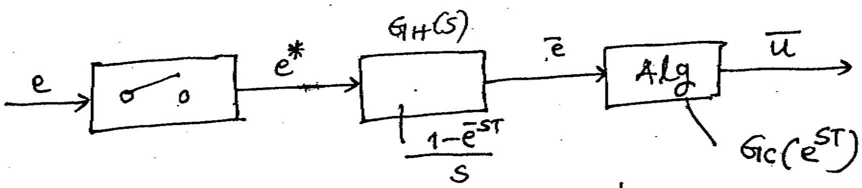
overall — controller (Algorithm) — system

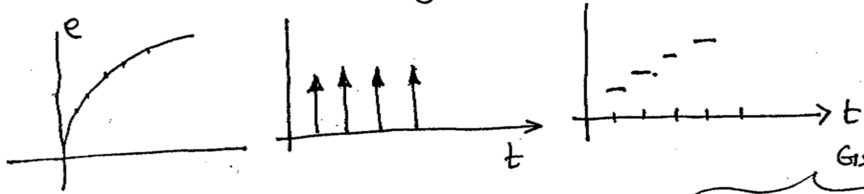
z

Lec-8









$$G_s(s) = \overbrace{\frac{b_0 + b_1 s + \dots + b_m s^m}{a_0 + a_1 s + \dots + a_n s^n}}^{G_{S_a}(s)} e^{-sT_d}$$

T_d := delay time $T_d = Td$, $d = 0, 1, 2, \dots$

Alg: PI controller : $G_c(e^{sT}) = \frac{\bar{U}(s)}{E(s)} = \frac{K_P + (K_I T - K_P) e^{-sT}}{1 - e^{-sT}}$

$$G_0(s) = G_H(s) . G_c(e^{sT}) . G_s(s) = \frac{X(s)}{E^*(s)}$$

$$G_0(s) = (1 - e^{-sT}) G_c(e^{sT}) . \frac{G_{Sa}(s)}{s} e^{-sdT}$$ delay

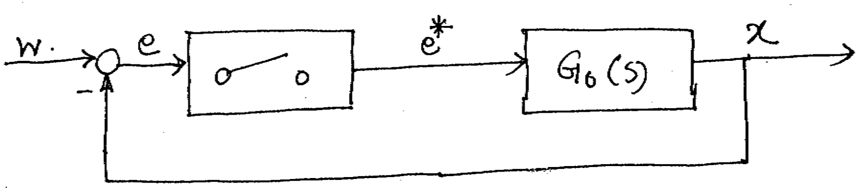
⇓
z domain

$$G_{0z}(z) = \mathcal{Z}\left\{ \underbrace{(1 - e^{-sT})\, e^{-sdT}\, G_c(e^{sT})}_{e^{sT} = z} . \frac{G_{Sa}(s)}{s} \right\}$$

overall z

$$G_{0z}(z) = (1 - z^{-1})\, G_c(z)\, z^{-d} . \mathcal{Z}\left\{ \frac{G_{Sa}(s)}{s} \right\}$$

$$\left[G_c(z) = \frac{K_P + (K_I T - K_P) z^{-1}}{1 - z^{-1}} \right]$$



$$X_z(z) = \mathcal{Z}\{G_0(s) \cdot E^*(s)\}$$

↓

$\mathcal{Z}$ function

$$= \mathcal{Z}\{G_0(s)\} \cdot \mathcal{Z}\{E^*(s)\}$$

$E_z(z)$ = is the $\mathcal{Z}$ transformation of $E^*(s)$

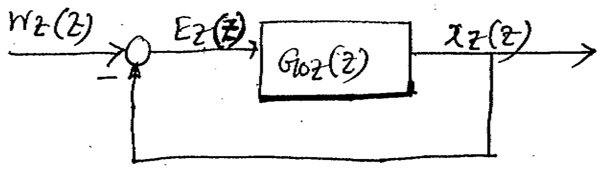
" " " " $e^*(t)$

" " " " $e(t)$

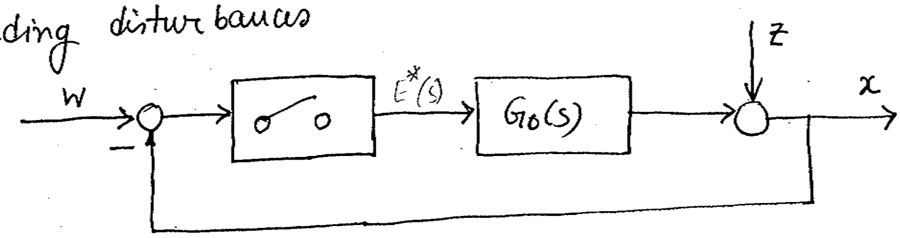
" " " " $E(s)$

$$X_z(z) = G_{0z}(z) \cdot E_z(z)$$

∴ form the last figure :

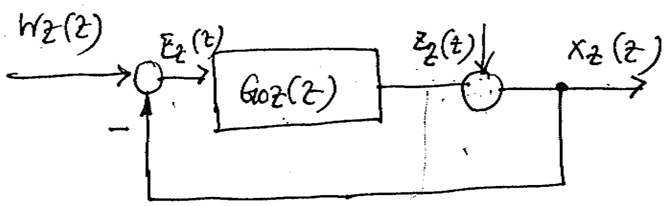


including disturbances



↓

$\mathcal{Z}$ domain



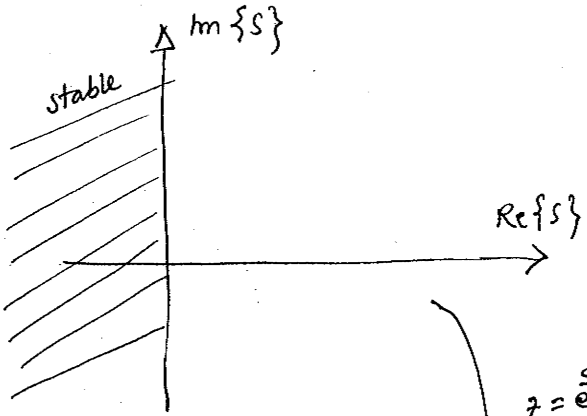
closed loop transfer function:

$$X_z(z) = Z_z(z) + G_{0z}(z)\left[W_z(z) - X_z(z)\right]$$

$$X_z(z) = \frac{G_{0z}(z)}{1 + G_{0z}(z)} W_z(z) + \frac{1}{1 + G_{0z}(z)} Z_z(z)$$

Stability in z domain:

stability in s domain is given if all poles are in left half s-plane;

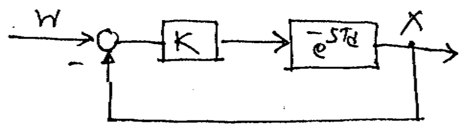


$s = j\omega$

$-\infty < \omega < \infty$

$z = e^{sT}$

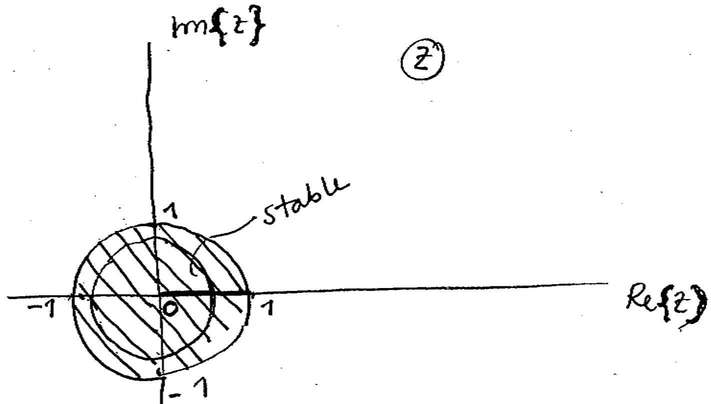
mapping



$G_0 \doteq K e^{-sT_d}$

$$\frac{X}{W} = G_{ce} = \frac{K e^{-sT_d}}{1 + K e^{-sT_d}}$$

Calculate the no. of poles for which K, it is stable?
also prove it for z domain



mapping of $s = j\omega$ $(\delta = 0)$ $\longrightarrow z = e^{j\omega T} = \cos\omega T + j\sin\omega T$

$|z| = 1$

left side: $s = \delta + j\omega \qquad \delta < 0$

$$z = e^{(\delta + j\omega)T} = \underset{<1}{e^{\delta T}} \cdot \underset{1}{e^{j\omega T}}$$

$$|z| = e^{\delta T} < 1$$

Mapping of the negative real axis: $s = \delta$

$z = e^{sT} \qquad \delta < 0$

from 1 to 0 in z domain

Example: PI controller $\quad G_C(z) = K_p \cdot \dfrac{z-1+\frac{K_I T}{K_e}}{z-1} = \dfrac{(z-a)}{z-1} K_p,$

$$a = 1 - \frac{K_I T}{K_p}$$

$$G(s) = \frac{K}{1+sT_1} e^{-sTd}$$

$T_d = Td$, $d=1$

$T_d = T$

$$G_0(s) = (1-e^{-sT}) e^{-sT} \quad G_C(e^{sT}) \cdot \frac{K}{(1+sT_1)s}$$

$$G_{0Z}(z) = (1-z^{-1}) z^{-1} \cdot K_p \cdot \frac{z-a}{z-1} \cdot \mathcal{Z}\left\{\frac{K}{(1+sT_1)s}\right\}$$

$$\frac{K}{s(1+sT_1)} = K\left(\frac{1}{s} - \frac{T_1}{1+sT_1}\right) = K\left(\frac{1}{s} - \frac{1}{s+\frac{1}{T_1}}\right)$$

$$\mathcal{Z}\left\{\frac{K}{(1+sT_1)s}\right\} = K\left(\mathcal{Z}\left\{\frac{1}{s}\right\} - \mathcal{Z}\left\{\frac{1}{s+\frac{1}{T_1}}\right\}\right) = K\left(\frac{z}{z-1} - \frac{z}{z-e^{-T/T_1}}\right)$$

$$G_{0Z}(z) = \frac{\cancel{(z-1)}(z-a)}{\cancel{z} \cdot z\,\cancel{(z-1)}} \cdot \frac{\cancel{z}\left(z-e^{-T/T_1}-(z-1)\right)}{(z-1)\,(z-e^{-T/T_1})} \; KK_p \qquad \frac{\cancel{z}-ze-\cancel{z}+z}{(z-1)\,(z-e^{-T/T_1})}$$

$$= \frac{(z-a)\,(1-e^{-T/T_1})}{z(z-1)(z-e^{-T/T_1})} KK_p$$

we can not change K, T_1 (It comes from the system)

a, K_p can be adjusted.

controller design:

compensation: $\quad a = e^{-\frac{T}{T_1}}$

then $\quad G_{0Z}(z) = \dfrac{1-e^{-T/T_1}}{z(z-1)} KK_p$

$$G_{0Z}(z) = K_0 \cdot \frac{1}{z(z-1)}$$

$$K_0 = KK_p\,(1-e^{-T/T_1})$$

closed loop: $\quad \dfrac{G_0(z)}{1+G_{0Z}} = \dfrac{X_Z(z)}{W_Z(z)} = \dfrac{K_0}{z(z-1)+K_0} = \dfrac{K_0}{z^2-z+K_0}$

08/07/2003

from previous class note:

$$G_{0Z}(z) = \frac{K_0}{z(z-1)}$$

closed loop: $\frac{G_0(z)}{1+G_0 z} = \frac{K_0}{z(z-1)+K_0} = \frac{K_0}{z^2-z+K_0}$

$$G_{wzyz}(z) = \frac{G_{0Z}(z)}{1+G_{0Z}(z)}$$

If disturbance is there, pole is same, so stability criteria is not changed.

$z^2-z+K_0=0$

$z_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4}-K_0}$

$\frac{1}{4}-K_0 \geq 0 \qquad K_0 \leq \frac{1}{4}$ real poles

$|z_{1/2}| \leq 1$ for stability!

for all $K_0 \leq \frac{1}{4}$

$\frac{1}{4}-K_0 \geq 0$

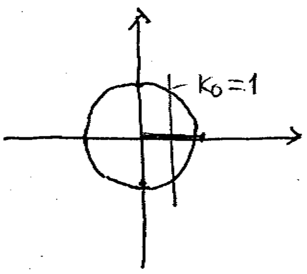
$\Rightarrow \frac{1}{4} \geq K_0$

$\Rightarrow K_0 \leq \frac{1}{4}$

for $K_0 > \frac{1}{4} \qquad z_{1/2} = \frac{1}{2} \pm j\sqrt{K_0-\frac{1}{4}}$

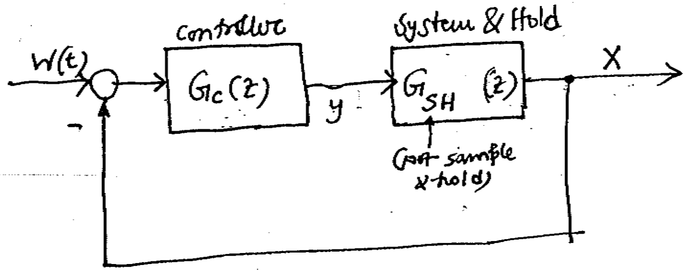
$|z_{1/2}| = \sqrt{\cancel{\tfrac{1}{4}}+K_0-\cancel{\tfrac{1}{4}}} = \sqrt{K_0} \leq 1 \Rightarrow$

System is stable for $0 < K_0 < 1$



Dead beat Controller Design:

Controller Design in z-domain!



steady state accuracy shall be reached in a finite time!

Known : $G_{SH}(z) = \boxed{\dfrac{b_1 z^{-1} + \cdots + b_n z^{-n}}{a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}} z^{-d}}$

represents delay e^{-sdT}

$d \to$ integer

$$G_{SH}(z) = \frac{X_Z(z)}{Y_Z(z)}$$

Limit value for Initial value : $\lim_{t \to 0} x(t) = x_0 = \lim_{z \to \infty} X_Z(z) = 0$

1) $d = 0$ (no delay time)

$$G_{wx}(z) = \frac{G_C(z)\, G_{SH}(z)}{1 + G_C(z)\, G_{SH}(z)} = \frac{X_Z(z)}{W_Z(z)}$$

closed loop

- assume: $w(t) = \sigma(t)$ $\quad W_Z(z) = \dfrac{z}{z-1}$

steady state accuracy after mT ; (m unknown now)

$x_k = w_k = 1 \quad$ for $k \geq m$

$y_{k+1} = y_k \quad k \geq m$

$$X_Z(z) = \underbrace{x_0 z^{-0}}_{0} + x_1 z^{-1} + \cdots + \underbrace{x_m z^{-m}}_{1} + \underbrace{x_m z^{-(m+1)}}_{x_{m+1} = x_m = 1} + \cdots$$

$$Y_Z(z) = y_0 + y_1 z^{-1} + \cdots + y_m z^{-m} + y_m z^{-(m+1)} + \cdots$$

$$G_{wx}(z) = \frac{X_Z(z)}{W_Z(z)} = \frac{z-1}{z} X_Z(z) = \left(1 - \frac{1}{z}\right) X_Z(z) = (1 - z^{-1}) X_Z(z)$$

$$= x_1 z^{-1} + x_2 z^{-2} + \cdots + 1 \cdot z^{-m} + z^{-(m+1)} + \cdots$$

$$- [\quad x_1 z^{-2} + x_{m-1} z^{-m} + 1 \cdot z^{-(m+1)} + \cdots$$

$$= x_1 z^{-1} + (x_2 - x_1) z^{-2} + \cdots + (1 - x_{m-1}) z^{-m}$$

$$= p_1 z^{-1} + p_2 z^{-2} + \cdots + p_m z^{-m} = P(z)$$

$$\boxed{\sum_{i=1}^{m} p_i = 1}$$

$$Gwy = \frac{y_z(z)}{w_z(z)} = (1-z^{-1})y_z(z)$$

$$= y_0 + y_1 z^{-1} + \cdots + y_m z^{-m} + y_m z^{-(m+1)} + \cdots$$

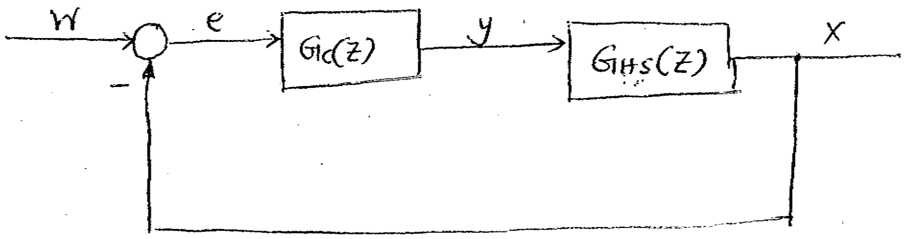
$$- [\; y_0 z^{-1} \qquad + y_{m-1} z^{-m} + y_m z^{-(m+1)} + \cdots \;]$$

$$= y_0 + (y_1 - y_0) z^{-1} + (y_2 - y_1) z^{-2} + \cdots (y_m - y_{m-1}) z^{-m}$$

$$\downarrow$$

$$q_0 + q_1 z^{-1} + q_2 z^{-2} + \cdots q_m z^{-m} = Gwy(z) = Q(z)$$

$$\boxed{\sum_{i=0}^{m} q_i = y_m}$$



$$G_{HS}(z) = \frac{b_1 z^{-1} + \dots + b_n z^{-n}}{a_0 + a_1 z^{-1} + \dots + a_n z^{-n}} \quad \underline{\text{known}}$$

$$G_{WX}(z) = \frac{G_C(z)\, G_{HS}(z)}{1 + G_C(z) G_{HS}(z)} = \frac{X_Z(z)}{W_Z(z)} \quad [\text{closed Loop}] \quad (*)$$

$$G_{WX}(z) = p_1 z^{-1} + \dots + p_m z^{-m} = P(z)$$

$$G_{WY}(z) = \frac{Y_Z(z)}{W_Z(z)} = q_0 + q_1 z^{-1} + \dots + q_m z^{-m} = Q(z)$$

Looking for $G_C(z)$!

from (*)

$$G_{WX}(z)\left[1 + G_C(z) G_{HS}(z)\right] = G_C(z)\, G_{HS}(z)$$

$$\Rightarrow G_C(z)\left[G_{WX}(z)\, G_{HS}(z) - G_{HS}(z)\right] = -G_{WX}(z)$$

$$\Rightarrow G_C(z) = \frac{G_{WX}(z)}{\left[1 - G_{WX}(z)\right] G_{HS}(z)}$$

$$G_{WX}(z) = P(z)$$

$$G_{HS}(z) = \frac{X_Z(z)}{Y_Z(z)} = \frac{\frac{X_Z(z)}{W_Z(z)}}{\frac{Y_Z(z)}{W_Z(z)}} = \frac{G_{WX}(z)}{G_{WY}(z)} = \frac{P(z)}{Q(z)} \quad (**)$$

$$\boxed{G_C(z) = \frac{P(z)}{\left[1 - P(z)\right] \frac{P(z)}{Q(z)}} = \frac{Q(z)}{1 - P(z)}}$$

From (**)

$$G_{HS}(z) = \frac{b_1 z^{-1} + \dots + b_n z^{-n}}{\underbrace{a_0 + a_1 z^{-1} + \dots + a_n z^{-n}}} = \frac{p_1 z^{-1} + \dots + p_m z^{-m}}{\underbrace{q_0 + q_1 z^{-1} + \dots + q_m z^{-m}}_{\text{unknown}}} = \frac{P(z)}{Q(z)}$$

Comparing the co-efficients:

If $G_{HS}(z)$ were normalized : $a_0 = 1$

$$G_{HS}(z) = \frac{b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} = \frac{1}{q_0} \frac{p_1 z^{-1} + \dots + p_m z^{-m}}{\left(1 + \frac{q_1}{q_0} z^{-1} + \dots \frac{q_m}{q_0} z^{-m}\right)}$$

$$b_1 = \frac{p_1}{q_0}$$

$$b_2 = \frac{p_2}{q_0}$$

$\vdots$

(1) $m = n$

(no. of samples until the steady state is reached)

number of samples until final values reached is equal to the order of the system.

$$b_n = \frac{p_n}{q_0}$$

$$\sum_{i=1}^{m} p_i = 1 \;\Rightarrow\; q_0 \sum_{i=1}^{m} b_i = 1$$

$$\therefore q_0 = \frac{1}{\sum_{i=1}^{m} b_i}$$

(1) $\boxed{p_1 = b_1 q_0 \;\dots\; p_n = b_n q_0}$

$$a_1 = \frac{q_1}{q_0} \;\dots\; a_n = \frac{q_m}{q_0} \;\Rightarrow\; \boxed{q_1 = a_1 q_0 ; \quad q_n = a_n q_0}$$

$\vdots$

(without delay time)

Now, we will include delay time to the system:

Delay time : $T_E = dT$, $d =$ integer

$$e^{-sT_t} = e^{-sdT}$$

$\downarrow$ z-domain

z^{-d}

$$G_{HS}(z) = \frac{b_1 z^{-1} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} z^{-d} \Bigg\} \quad G_{HS}(z) = \frac{\bar{b}_1 z^{-1} + \bar{b}_2 z^{-2} \dots + \bar{b}_{1+d} z^{-(1+d)} + \dots + \bar{b}_{n+d} z^{-(n+d)}}{1 + \bar{a}_1 z^{-1} + \dots + \bar{a}_n z^{-n} + \dots + \bar{a}_{n+d} z^{-(n+d)}}$$

$$\overline{b_1} = 0$$
$$\vdots$$
$$\overline{b_d} = 0$$
$$\overline{b_{1+d}} = \underline{b_1}$$
$$\vdots$$
$$\overline{b_{n+d}} = b_n$$

$$\overline{a_1} = a_1$$
$$\vdots$$
$$\overline{a_n} = a_n.$$
$$\overline{a_{n+1}} = 0$$
$$\vdots$$
$$\overline{a_{n+d}} = 0$$

Lec-11

22/07/2003

Example: $G(s) = \frac{K}{1+sT_s} e^{-T_t s}$ ← delay time (T_t); T_s → time const.

$T_t = d \cdot T$

$G_{HS}(s) = \frac{1-e^{-sT}}{s} \cdot \frac{K}{1+sT_s} e^{-T_t s}$

↓ z-transformation

$G_{HSZ}(z) = \underbrace{(1-z^{-1})}_{\frac{z-1}{z}} z^{-d} \mathcal{Z}\left\{\frac{K}{s(1+sT_s)}\right\}$

K গেলো কই?

$\frac{(1-c)\cancel{z}}{(\cancel{z-1})(z-c)} \bigg| \; c = e^{-\frac{T}{T_s}}$

$= \frac{z^{-d}(1-c)}{z-c}$

$= \frac{\overbrace{(1-c)}^{b_1} z^{-1}}{\underset{a_0}{1} - \underbrace{cz^{-1}}_{a_1}} z^{-d}$ $\qquad z(1-cz^{-1})$

$G_{HSZ}(z) = \frac{\overbrace{\bar{b}_1 z^{-1} + \dots}^{a_0} + \overbrace{(1-c)}^{\bar{b}_{d+1}} z^{-(d+1)}}{1-cz^{-1}}$ $\qquad \bar{b}_d \, z^{0+d}$

$\bar{b}_1 = 0$

$\vdots$

$\bar{b}_d = 0$

$b_{d+1} = b_1 = 1-c$

$a_0 = 1$

$a_1 = -c$

$a_2 = 0$

$\vdots$

$a_{d+1} = 0$

$q_0 = \left(\sum_{i=1}^{d+1} b_i\right)^{-1} = \frac{1}{1-c}$

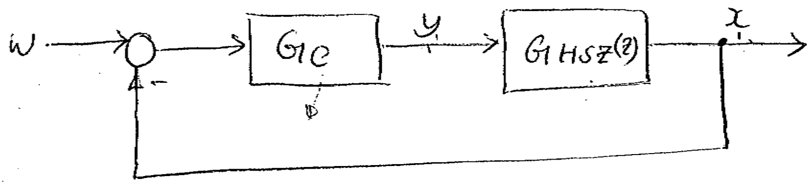
$q_1 = a_1 q_0 = \frac{-c}{1-c}$ $\qquad q_2 = q_3 = q_{d+1} = 0$

$p_{1+d} = b_1 q_0 = (1-c) \cdot \frac{1}{1-c} = 1$

↓

$\bar{b}_{d+1}$

$$G_C(z) = \frac{Q(z)}{1-P(z)} = \frac{q_0 + q_1 z^{-1}}{1 - 1 \cdot z^{-(1+d)}} = \underline{\underline{\frac{1}{1-c} \cdot \frac{1-cz^{-1}}{1-z^{-(1+d)}}}}$$



$$G_{wy}(z) = \frac{y_Z(z)}{W_Z(z)} = Q(z)$$

$$y_Z(z) = W_Z(z) \cdot Q(z)$$

$$y_Z(z) = W_Z(z) \left(\frac{1}{1-c} - \frac{c}{1-c} z^{-1} \right)$$

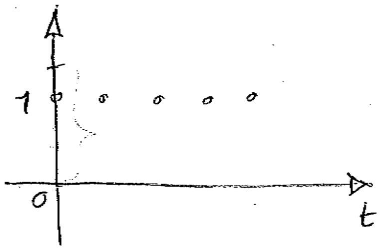
$\downarrow$ time

$W_0 = 1$

$$y_K = \frac{1}{1-c} W_K - \frac{c}{1-c} W_{K-1}$$

$\uparrow$ shift

(W is a step function)



$$K=0: \ y_0 = \frac{1}{1-c} W_0 = \frac{1}{1-c}$$

$$K=1: \ y_1 = \frac{1}{1-c} \cdot 1 - \frac{c}{1-c} \cdot 1 = 1$$

$$K=2: \ y_2 = \frac{1}{1-c} \cdot 1 - \frac{c}{1-c} \cdot 1 = 1$$

only c has the influence on T (Sampling time)

$$c = e^{-T/T_S} \qquad c = \frac{1}{e^{\frac{T}{T_S}}}$$

when, $T \downarrow$, $c \uparrow$

$\therefore y_0 \rightarrow \uparrow$ (y_0 becomes higher, when T becomes smaller)

$$G_{wx}(z) = \frac{x_Z(z)}{W_Z(z)} = P(z) = z^{-(1+d)}$$

$$x_Z(z) = W_Z(z) \cdot P(z) = z^{-(1+d)} \cdot W_Z(z)$$

W_{K-3}

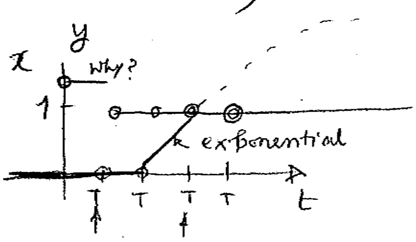
$\downarrow$

$x_K = W_{K-(1+d)}$ Assume, $d = 2$: $x_0 = 0$ ($K=0$)

$x_1 = 0$

$x_2 = 0$

$x_3 = 1$

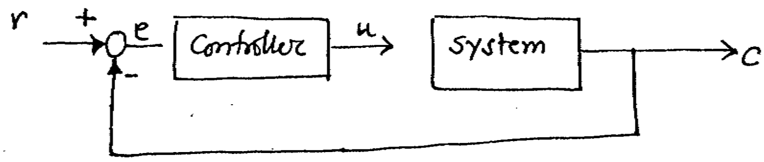


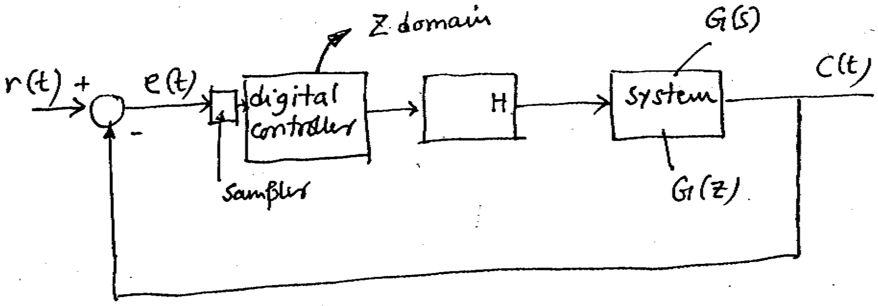
Lec-12

(Last lecture)

Martin Ruel

29/07/2003





⇒ develop: $G_C(z)$

→ $G(z)$ together with S&H

$$G_{S\&H}(s) = \frac{1-e^{-sT}}{s}$$

sampling time

$$G_{S\&H}(z) = \frac{z-1}{z} = (1-z^{-1})$$

<u>Example:</u>

$$G(s) = \frac{(s-1)(1-e^{-sT})}{s(s+1)^2(1+e^{-sT})}$$

$G(z) = ?$

first of all, $z = e^{sT}$

$$G(s) = \mathcal{Z}\left\{\frac{s-1}{s(s+1)^2}\right\} \cdot \mathcal{Z}\left\{\frac{1-e^{-sT}}{1+e^{-sT}}\right\}$$

$$= \mathcal{Z}\left\{\frac{s-1}{s(s+1)^2}\right\} \cdot \frac{1-z^{-1}}{1+z^{-1}}$$

$$= \mathcal{Z}\left\{\frac{s-1}{s(s+1)^2}\right\}\left(\frac{z-1}{z+1}\right)$$ (multiplying by z)

$$\mathcal{Z}\left\{\frac{s-1}{s(s+1)^2}\right\} = \mathcal{Z}\left\{\frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{(s+1)^2}\right\}$$

$$A_1 = \lim_{s\to 0} \frac{s-1}{(s+1)^2} = \frac{-1}{1} = -1$$

$$A_2 = \lim_{s\to -1} \frac{d}{ds}\left[\frac{(s-1)}{s}\right] = \lim_{s\to -1} \frac{d}{ds}\left[\frac{s-1}{s}\right] = \lim_{s\to -1} \frac{s(1-0)-(s-1)\cdot 1}{s^2}$$

$$= \lim_{s\to -1} \frac{s-s+1}{s^2} = \frac{1}{1} = 1$$

$A_2 = 1$ pls. see

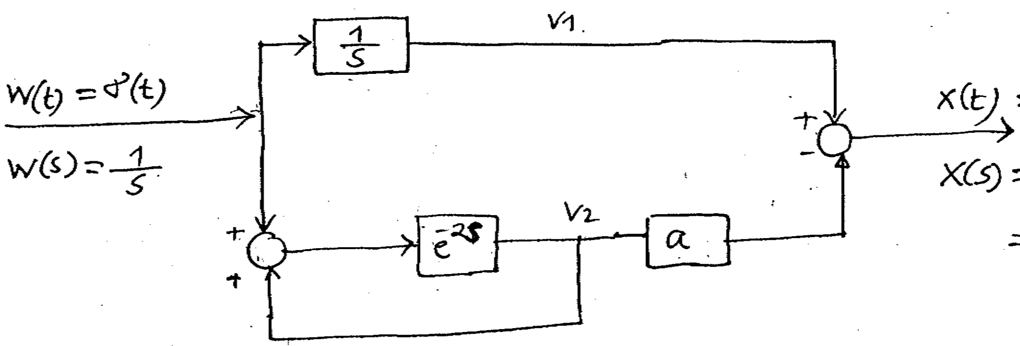
$$A_3 = \lim_{s\to -1} \frac{s-1}{s} = \frac{-1-1}{-1} = 2$$

$$z\left\{\frac{s-1}{s(s+1)^2}\right\} = z\left\{\frac{A_1}{s}+\frac{A_2}{s+1}+\frac{A_3}{(s+1)^2}\right\}$$

$$= z\left\{-\frac{1}{s}+\frac{1}{s+1}+\frac{2}{(s+1)^2}\right\}$$

$$= -\frac{z}{z-1}+\frac{z}{z-e^{-T}}+\frac{2Tze^{-T}}{(z-e^{-T})^2}$$

$$= \frac{-z(z-e^{-T})^2+2T(z-1)ze^{-T}+z(z-1)(z-e^{-T})}{(z-1)(z-e^{-T})^2}$$



$$X(S) = W(S).G(S)$$
$$= \frac{1}{s}\left(\frac{1}{s}-a\cdot\frac{e^{-2s}}{1-e^{-2s}}\right)$$

$$X(z) = Z\left\{\frac{1}{s^2}\right\}-aZ\left\{\frac{1}{s}\right\}\cdot Z\left\{\frac{e^{-2s}}{1-e^{-2s}}\right\} \qquad z=e^{sT}$$

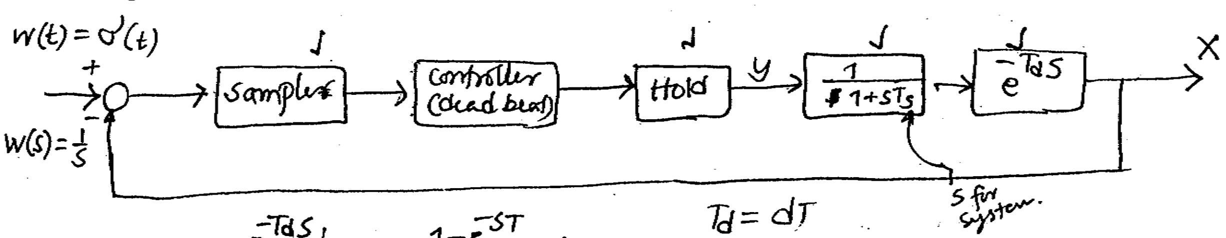
$$= \frac{Tz}{(z-1)^2}-a\frac{z}{z-1}\cdot\frac{z^{-\frac{2}{T}}}{1-z^{-\frac{2}{T}}} \qquad e^{-2s} = e^{(sT)\cdot-\frac{2}{T}}$$

Controller Design:

— Quasi-continuous Controller

$$PID: y_z(z) = K_p\left(e(z)+\frac{T_v}{T}\cdot\frac{z-1}{z}e(z)+\frac{T}{T_n}\frac{z}{z-1}e(z)\right)$$

— Dead beat Controller



$$T_d = dT$$

$$G_0(s) = \frac{e^{-T_d s}}{1+sT_s}\cdot\frac{1-e^{-sT}}{s}$$

$$G_0(z) = Z\left\{\frac{1}{s(1+sT_s)}\right\}Z\left\{e^{-dTs}(1-e^{-sT})\right\}$$

$$= Z\left\{\frac{1}{s(1+sT_s)}\right\}\left(z^{-d}(1-z^{-1})\right)$$

$$= Z\left\{\frac{\frac{1}{T_s}}{s\left(\frac{1}{T_s}+s\right)}\right\}(1-z^{-1})z^{-d}$$

from table:

$$Z\left\{\frac{a}{s(s+a)}\right\} = \frac{(1-C)\,z}{(z-1)(z-C)} \quad ; C = e^{-aT}$$

$$\therefore \quad Z\left\{\frac{\frac{1}{T_S}}{s(s+\frac{1}{T_S})}\right\} = \frac{(1-C)z}{(z-1)(z-C)} \quad ; C = e^{-\frac{T}{T_S}}.$$

$$\therefore G_0(z) = \frac{(1-C)z}{\cancel{(z-1)}(z-C)} \times \frac{\cancel{(z-1)}}{z} \cdot z^{-d}$$

$$= \frac{(1-C)\,z^{-1}}{1-Cz^{-1}}\, z^{-d}$$

$$= \frac{b_1 z^{-1}}{1+a_1 z^{-1}}\, z^{-d}$$

$b_1 = 1-C \quad ; \quad b_2 = b_3 = \cdots = 0$

$a_1 = -C \quad ; \quad a_2 = a_3 \cdots = 0$

$$q_0 = \frac{1}{\sum b_i} = \frac{1}{1-C}$$

$$q_1 = a_1 \cdot q_0 = -\frac{C}{1-C}$$

$$q_2 = a_2 q_0 = 0$$

$$\underset{\downarrow \atop b_{1+d}}{P_{1+d}} = b_1 \cdot q_0 = (1-C)\cdot\frac{1}{(1-C)} = 1$$

$$P_{2+d} = b_2 q_0 = 0$$

$$G_c(z) = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - P_{1+d}\, z^{-(1+d)} - P_{2+d}\, z^{-(2+d)}} = \frac{Q(z)}{1-P(z)}$$

$$= \frac{\frac{1}{1-C} - \frac{C}{1-C} z^{-1} + 0}{1 - z^{-(1+d)} - 0} \qquad C = e^{-\frac{T}{T_S}}$$

(depends on system & sample time) that is important.

$$= \frac{1}{1-C} \cdot \frac{1 - Cz^{-1}}{1 - z^{-(1+d)}}.$$

$$\cancel{W}\; Y(z) = W(z)\, G_{WY}(z) = Q(z)\cdot W(z)$$

$$= W(z)\cdot \frac{1}{1-C}\,(1 - C z^{-1})$$

$$Q(z) = q_0 + q_1 z^{-1}$$

$$Y(TK) = \frac{W(TK)}{1-C} - \frac{C}{1-C}\, W(TK-1)$$

$$W(z)\, z^{-1}$$

$$X(z) = W(z)\, W_{WX}(z) = W(z)\, P(z)$$

$$= W(z)\, P_{1+d}\, z^{-(1+d)} = W(z)\, z^{-(1+d)}$$

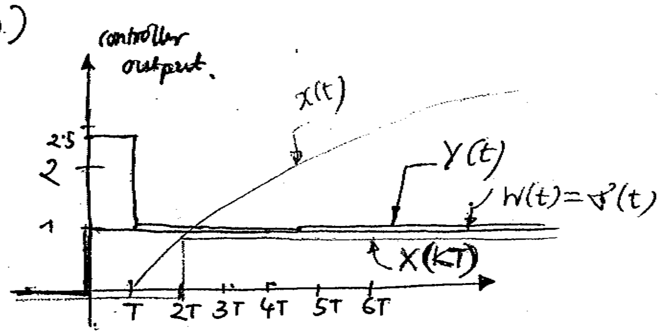
$$\Downarrow$$

$$X(K) = W(k-(1+d))$$

$d = 1$

$T_d = T$

$T_S = 2T_d = 2T$

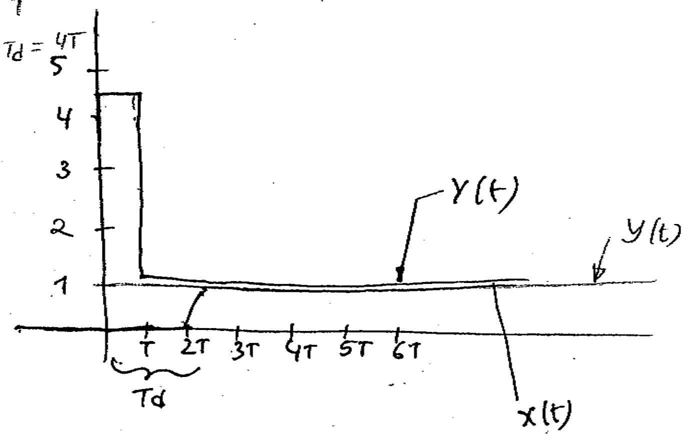


$d = 2$

$T_d = T$

$T_S = 2T_d = 4T$

$\therefore \frac{1}{1-C} = 4.52$



output with system(?)

delaytime is a part of the system,