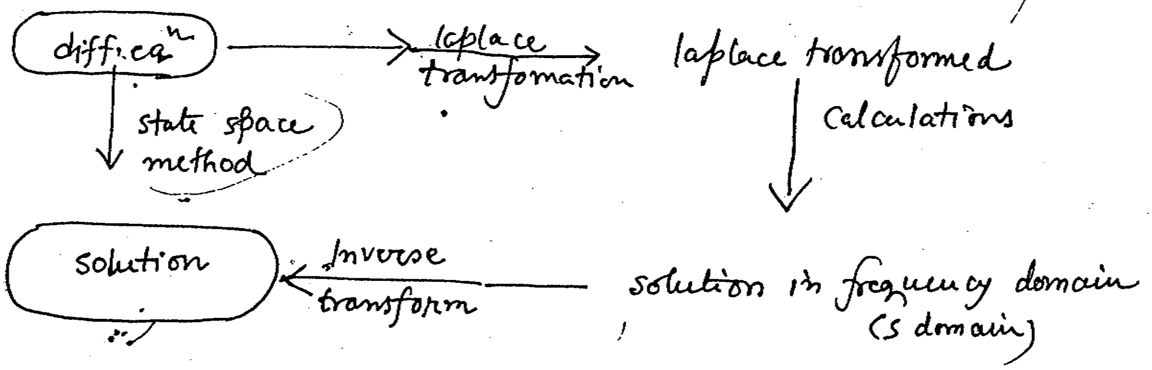
Advanced Control 1

22/10/2002

Prof. Mayer

Previous analysis of closed loops:



Earlier Exclusively frequency domain methods
since the beginning of the 60's; also state space methods

- strategy in the case of state space methodology:
  * one remains in the time domain
  * representation via systems of diff. eqn's
  * descriptions through state variables from the state space
  * the entirely of the state variables from the state space

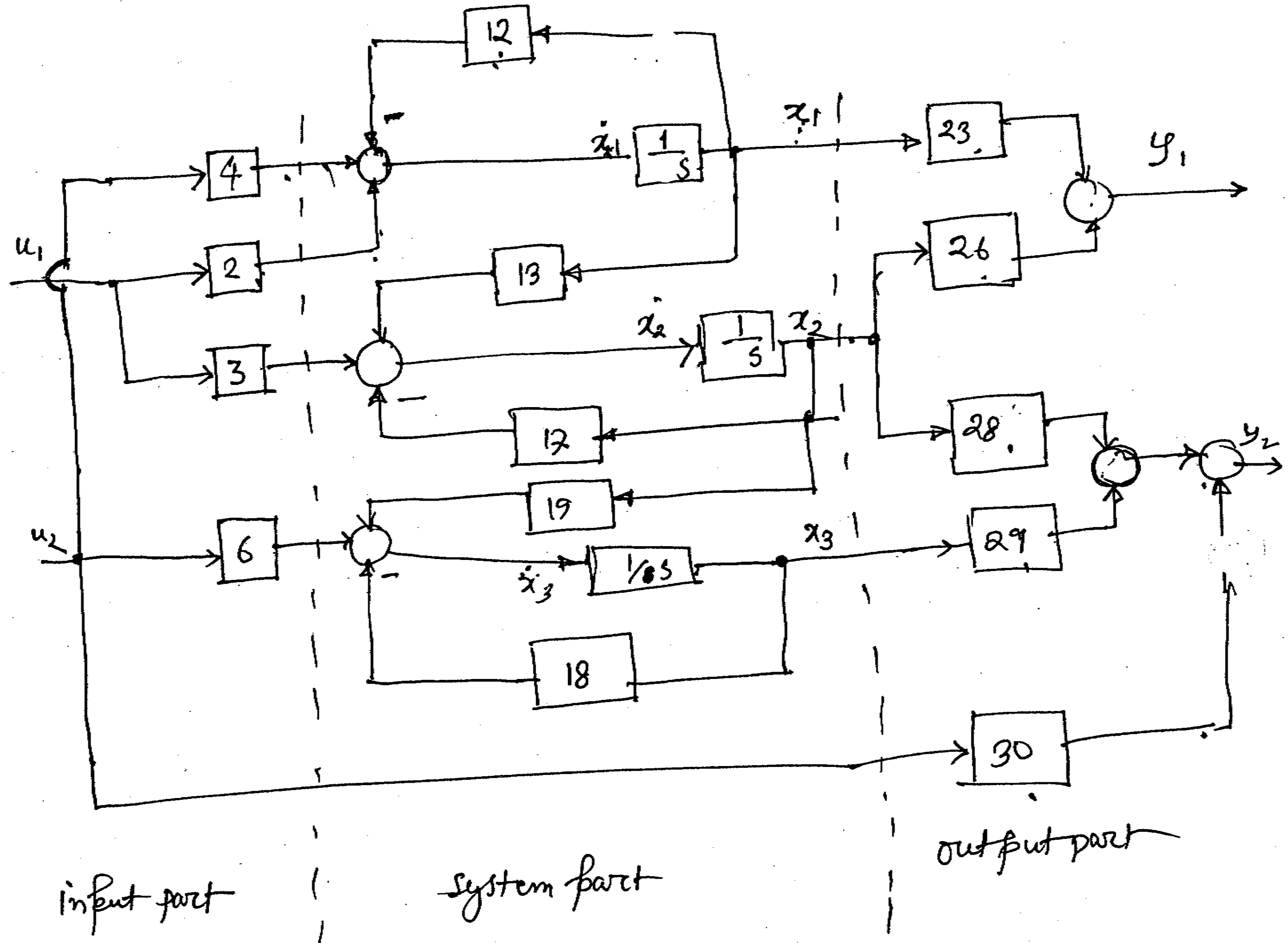
- Advantages of the state space representation:
  * Applicability also on non linear and time variant systems
  * further insights into the system behaviour are available
  * internal processes can be more precisely observed
  * efficient treatment for multivariable systems
  * easy transferability to numeric computation methods on digital computers

It depends on the particular tasks, whether it is more advantageous to perform the analysis of control circuits either in the frequency domain or in the state ~~of~~ space description.

Time domain
state space description

Transformation into state space description

Example : linear time invariant multivariable system



$s \rightarrow$ complex frequency

Setting up the differential eqⁿs:

$$\dot{x}_1 = -12x_1 + 2u_1 + 4u_2$$
$$\dot{x}_2 = 13x_1 - 17x_2 + 3u_1 \quad 0$$
$$\dot{x}_3 = 19x_2 - 18x_3 + 6u_2 \quad 0 \quad 6$$

Output equations:

$$y_1 = 23x_1 + 26x_2$$
$$y_2 = 28x_2 + 29x_3 + 30u_2$$

$n \times p$ c columns, rows

System altogether is of 3rd order

3x1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -12 & 0 & 0 \\ 13 & -17 & -17 \\ 0 & 19 & -18 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}(t)} + \underbrace{\begin{bmatrix} 2 & 4 \\ 3 & 0 \\ 0 & 6 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ - \end{bmatrix}}_{\underline{u}(t)}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 23 & 26 & 0 \\ 0 & 28 & 29 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 30 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\underline{u}(t)}$$

Introduction of the vectors :

state vectors:

state vector :

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \dim[n]$$

input vector

$$\underline{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix} \qquad \dim[p]$$

output vector :

$$\underline{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{bmatrix} \qquad \dim[q]$$

If the input variables $\underline{u}(t)$ for $t > t_0$ as well as the initial state $\underline{x}(t_0)$ are present, the intermediate values $\underline{x}(t)$ for $t > t_0$ are uniquely definite.

t_0 is any date

The system behavior for before $t = t_0$ plays not any role. It is without importance, in which way the initial values $\underline{x}(t_0)$ come up.

Generation'l Notation:

square matrix

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}}_{\dot{\underline{x}}(t)} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{\underline{A}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\underline{x}(t)}, +$$

$$+ \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}}_{\underline{B}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}}_{\underline{u}(t)}$$

$$\underbrace{\begin{bmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{bmatrix}}_{\underline{y}(t)} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{q1} & \cdots & c_{qn} \end{bmatrix}}_{\underline{C}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\underline{x}(t)} + \underbrace{\begin{bmatrix} d_{11} & \cdots & d_{1p} \\ \vdots & & \vdots \\ d_{q1} & \cdots & d_{qp} \end{bmatrix}}_{\underline{D}} \underbrace{\begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}}_{\underline{u}(t)}$$

→ brief notation →

State Equations:

$$\boxed{\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}\cdot\underline{x}(t) + \underline{B}\cdot\underline{u}(t) \\ \underline{y}(t) &= \underline{C}\cdot\underline{x}(t) + \underline{D}\cdot\underline{u}(t) \end{aligned}}$$

Representation in state space

Denotation:

upper eq^n : "state differential equation"

lower eq^n : " output eq^n "

both eq^{n}s together : "state eq^{n}s"

Representation in state space

Denotation matrices:

$\underline{A}$ " evolution matrix " $\dim[\underline{A}] = n \times n$

The evolution matrix is a square matrix

$\underline{B}$ " control matrix" $\dim[\underline{B}] = n \times p$

$\underline{C}$ " observation matrix" $\dim[\underline{C}] = q \times n$

$\underline{D}$ " Direct transmission matrix " $\dim[\underline{D}] = q \times p$

In case of linear time invarient systems the matrices $\underline{A}$, $\underline{B}$, C and $\underline{D}$ consist exclusively of constant elements

Important:

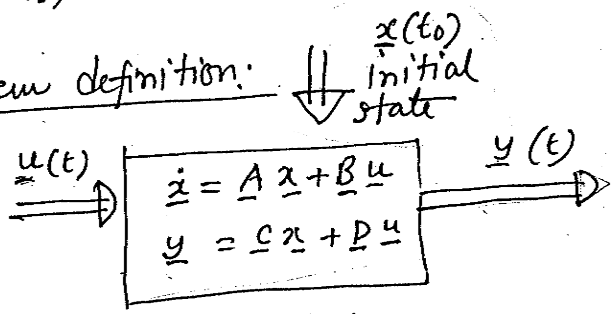
The input vector $\underline{u}(t)$ consists only of the input variables, but not of their derivatives.

$\Rightarrow$ the state variables in $\underline{x}(t)$ have no irregularities

Annotation:

Thereby it is assumed that at the system entry step functions, but never δ- impulse functions may be assigned.

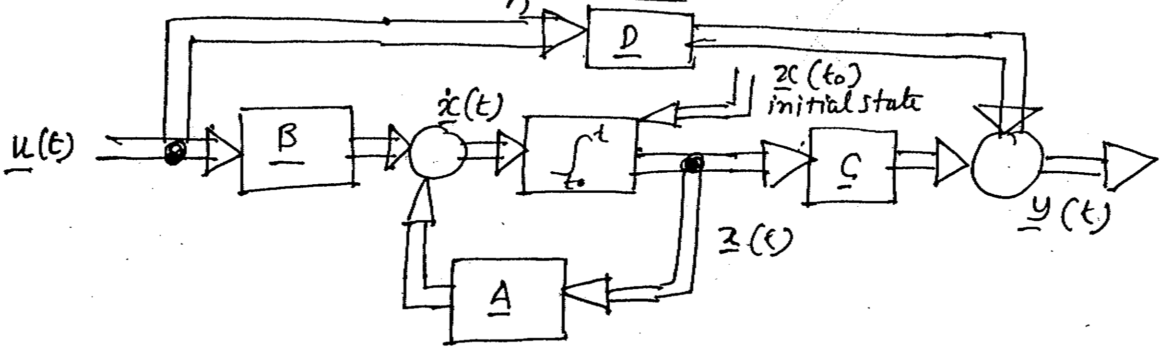
System definition:



double lines clarify the transfer of vector variables.

Special case: In state space description, which is pointed out here, time delay elements can not be taken into consideration.

Flow digrams of the state equations:



In practice the following applies:

- $p \leq n$: The number n of state variables in most cases is considerably higher than the number p of input variables.
- $q = p$: furthermore, the number q of output variables in most cases is equal to the number p of input variables.
- $\underline{D} = \underline{0}$: As input variables act normally only indirectly via the state vector $\underline{x}$ on the output variables, here the direct transmission matrix consists exclusively of zero elements.

- State variables
- input variables
- output variables

Advanced Control 1

Lec-2

Contd.

Example: Single input/output system of a third order system without direct signal transmission.

It applies: $n > 1$ (Here $n=3$)

$q = p = 1$

$\underline{D} = \underline{0}$

$$\dot{\underline{x}}(t) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \underline{x}(t)$$

It results: $\underline{B} \longrightarrow \underline{b}$ lower case

$\underline{C} \longrightarrow \underline{c}^T$

upper case

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad \underline{c}^T = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

(Capital letters for matrices; lower case for column vectors)
This does not apply to signals!

Brief notation:

$$\begin{cases} \dot{\underline{x}} = \underline{A} \cdot \underline{x} + \underline{b}\,\underline{u} \\ y = \underline{c}^T \cdot \underline{x} \end{cases}$$

in practice often occurring form of the state eq^{n}s

22 (b) Derivation of the state equations from a linear differential equation of higher order:

Example: Differential equation of second order

$$\ddot{y}(t) + c_2 \dot{y}(t) + c_1 y(t) = c_0 u(t)$$

Introduction of state variables: $x_1(t) = y(t) \rightarrow \boxed{\dot{x}_1(t) = \dot{y}(t)}$

$x_2(t) = \dot{y}(t) \rightarrow \dot{x}_2(t) = \ddot{y}(t)$

$$\dot{x}_1(t) = x_2(t) \quad \rightarrow$$

$$\dot{x}_2(t) = -c_2 x_2(t) - c_1 x_1(t) + c_0 u(t) \rightarrow$$

(from the above two differential eq^{n}s)

in vector notation

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -c_1 & -c_2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ c_0 \end{bmatrix} u(t)$$

Output eqⁿ:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(t)$$

A differential equation of second order is transformed into a system of 2 differential equations of first order and the output equation.

The transfer Matrix

Multi variable system: Transfer matrix

Single input/output system: transfer function

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$

$$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t)$$

! Transformation into the frequency domain

(for signals: capital letters for variables transformed in the frequency domain)

$$s\,\underline{X}(s) = \underline{A}\,\underline{X}(s) + \underline{B}\,\underline{U}(s) \qquad \text{with initial values } \underline{x}(0) = \underline{0}$$

$$\underline{Y}(s) = \underline{C}\,\underline{X}(s) + \underline{D}\,\underline{U}(s)$$

$$(s\underline{I} - \underline{A})\,\underline{X}(s) = \underline{B}\,\underline{U}(s)$$

$$\underline{X}(s) = (s\underline{I} - \underline{A})^{-1}\,\underline{B}\,\underline{U}(s)$$

$$\underline{Y}(s) = \underbrace{\left[\underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} + \underline{D}\right]}_{\underline{G}(s)}\underline{U}(s)$$

The transfer matrix reads:

$$\boxed{\underline{G}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} + \underline{D}}$$

Special case: single input/output system

$$G(s) = \underline{c}^T(s\underline{I} - \underline{A})^{-1}\,\underline{b} + d$$

Example: Single input/output system of 2nd order

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\underline{c}^T = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \qquad d = d$$

$$G(s) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s-a_{11} & -a_{12} \\ -a_{21} & s-a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + d$$

$$= \frac{\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s-a_{22} & a_{12} \\ a_{21} & s-a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}{s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}} + d$$

Numerical values : $a_{11} = -1,\ a_{12} = 2,\ b_1 = 3,\ c_1 = 2$

$a_{21} = -4,\ a_{22} = -6,\ b_2 = 5,\ c_2 = 7$

$d = 0$

$$G(s) = \frac{41s + 7}{s^2 + 7s + 14}$$

$d = 3$

$$G(s) = \frac{3 \cdot s^2 + 62s + 49}{s^2 + 7s + 14}$$

(If $d > 0$, order of numerator = order of denominator)

Conclusion: $d \neq 0$ increases the order of the numerator by compared to $d = 0$

For multivariable systems, instead of the complex transfer function $G(s)$ the complex transfer matrix $\underline{G}(s)$ exist.

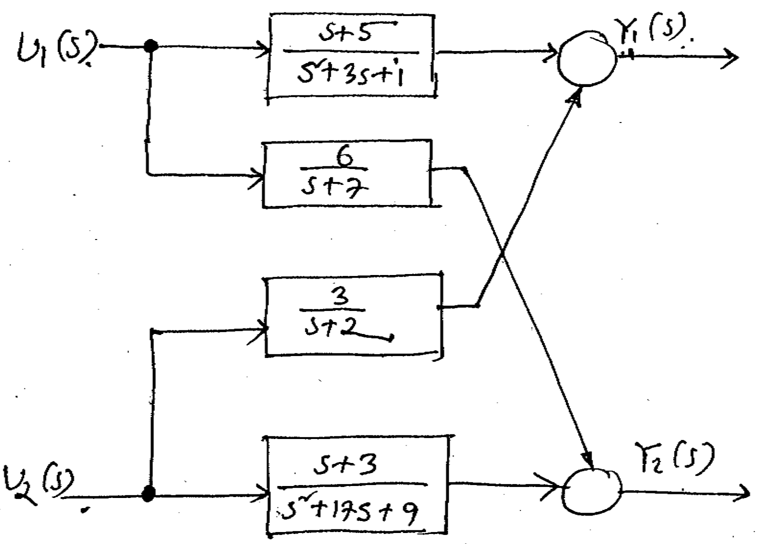
Example: system with 2 input and 2 output variables

$$\underline{G}(s) = \begin{bmatrix} \frac{s+5}{s^2+3s+1} & \frac{3}{s+2} \\ \frac{6}{s+7} & \frac{s+3}{s^2+17s+9} \end{bmatrix}$$

$$\underline{Y}(s) = \underline{G}(s) \cdot \underline{U}(s)$$

$$Y_1(s) = \frac{s+5}{s^2+3s+1}\, U_1(s) + \frac{3}{s+2} \cdot U_2(s)$$

$$Y_2(s) = \frac{6}{s+7} \cdot U_1(s) + \frac{s+3}{s^2+17s+9} \cdot U_2(s)$$



Controllability & observability

Controllability

The linear time invariant system

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$

is mentioned as controllable, if during a finite time period $t_0 \leq t \leq t_1$ each initial state x_0 at the moment t_0 can be moved to any other state $\underline{x}_1$ at the moment by a suitable choice of the control vector, $\underline{u}(t)$

Here are:

* trajectories $\underline{x}(t)$ not predetermined
* no limitation of the control vector

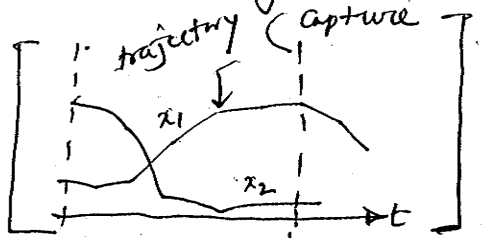
Determination of the controllability

controllability matrix:

$$\underline{Q}_c = [\underline{B}, \underline{A}\underline{B}, \cdots \underline{A}^{n-1}\underline{B}]$$ (This is a square matrix)

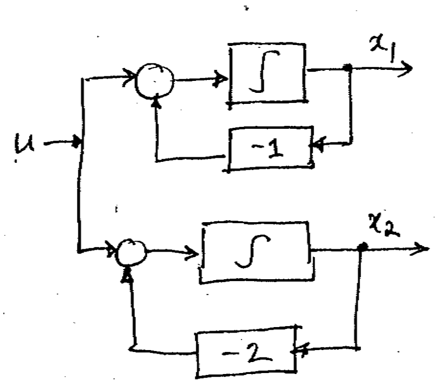
Controllability condition:

A linear timeinvariant system of n-th order is then and only then controlleble, if the controllability matrix has the rank n.



Example

1) $\underline{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Here $n=2$



$$Q_c = \begin{bmatrix} \underline{b} & \underline{A}\,\underline{b} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$ rank $Q_c = 2$

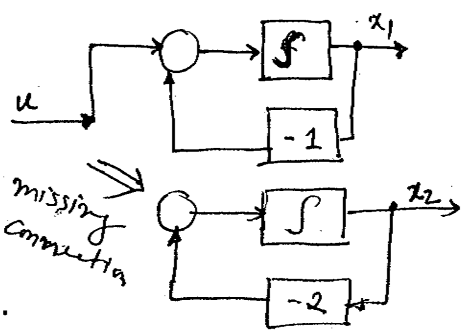
System is controllable

2) $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$Q_c = [b, \underline{A}\,\underline{b}]$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$ rank $Q_c = 1$

System is not controllable.

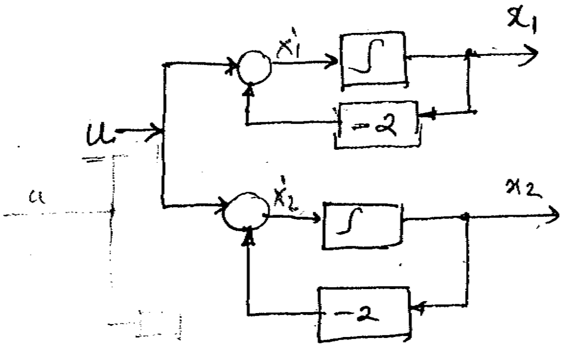


3) $\underline{A} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{Q}_c = [\underline{b}, \underline{A}\underline{b}]$ rank $\underline{Q}_c = 1$

$\therefore$ System is not controllable

$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$



$\dot{x}_1 = -2x_1 + 1 u$
$\dot{x}_2 = -2x_2 + 1 u$

Observability:

The linear time invariant system

$$\dot{x}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t)$$

is mentioned as observable, if with known $\underline{u}(t)$ from the measurement of $\underline{y}(t)$ during a finite time period $t_0 \leq t \leq t_1$ the initial state $\underline{x}_0$ at the moment t_0 can be clearly determined.

Here it is important, where the initial state $\underline{x}_0$ is located.

Determination of the observability:

Observability matrix;

$$Q_o = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T A \\ \vdots \\ \underline{c}^T A^{n-1} \end{bmatrix}$$

A linear time invariant system of n-th order is then and only then observable if the matrix $\underline{Q}_o$ has the rank n

Example:

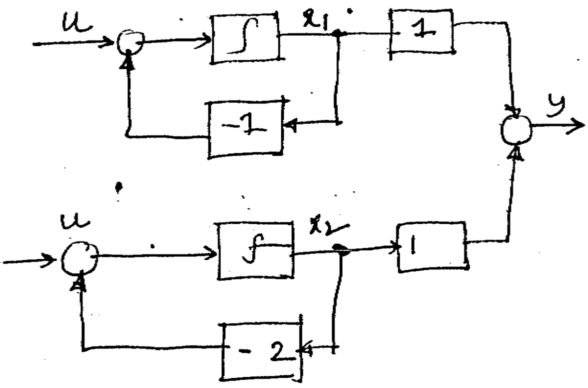
1) $\underline{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

$\underline{c}^T = [1 \quad 1]$

$\underline{Q}_o = \begin{bmatrix} c^T \\ c^T A \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

rank $Q_o = 2$

System is observable

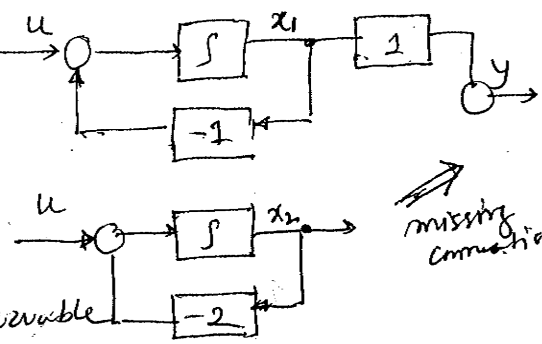


2) $\underline{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

$\underline{c}^T = [1 \quad 0]$

$\underline{Q}_o = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ rank $Q_o = 1$

System is not observable

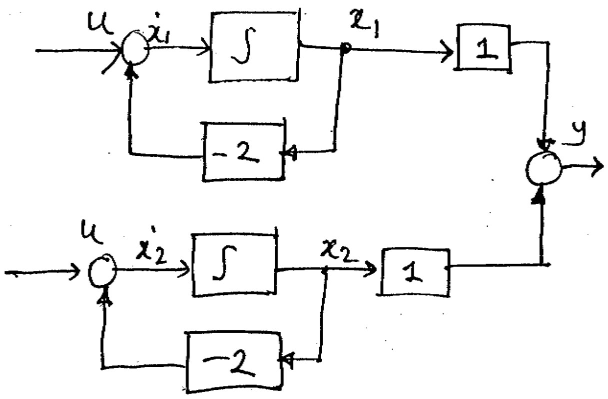


missing connection.

3) $\underline{A} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$\underline{c}^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$\underline{Q}_0 = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$ rank $\underline{Q}_0 = 1$



we will evaluate some examples

<u>internet</u>

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Exercises

Lec-3.
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The controllability canonical form:

Linear time invariant system of nth order

Transfer function into the frequency domain:

$$Y(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots a_1 s + a_0} U(s) = \frac{Z(s)}{N(s)} U(s).$$

Corresponding differential eq^{n}s

$$y^{(n)} + a_{n-1} y^{(n-1)} \dots + a_1 \dot{y} + a_0 y = b_0 u + b_1 \dot{u} + \dots + b_{n-1} u^{(n-1)} + b_n u^{(n)}$$

disadvantage: Derivatives of $u(t)$ on the right hand side of the eq^n.

Substituting in the frequency domain

$$X_1 = \frac{1}{N(s)} U$$

$$X_2 = \frac{s}{N(s)} U \qquad sX_1 = X_2 \qquad sX_1 = X_2 \qquad \circ \longrightarrow \circ \quad \dot{x}_1 = x_2$$

$$X_3 = \frac{s^2}{N(s)} U \qquad sX_2 = X_3 \qquad s^2 X_1 = X_3 \qquad \circ \longrightarrow \circ \quad \dot{x}_2 = x_3$$

$$X_n = \frac{s^{n-1}}{N(s)} U \qquad sX_{n-1} = X_n \qquad s^{n-1} X_1 = X_n \qquad \circ \longrightarrow \circ \quad \dot{x}_{n-1} = x_n$$

$$s^n X_1 = sX_n \qquad \dot{x}_n = (\qquad\qquad) \dots (a)$$

The initial values are assumed as zero

Calculation of $\dot{x}_n$

from $X_1 = \frac{1}{N(s)} U$ follows $\rightarrow N(s) X_1 = U$

$$\underbrace{s^n X_1}_{sX_n} + a_{n-1} \underbrace{s^{n-1} X_1}_{X_n} + \dots + a_1 \underbrace{sX_1}_{X_2} + a_0 \underbrace{X_1}_{X_1} = U$$

$$sX_n = -a_{n-1} X_n \dots - a_1 X_2 - a_0 X_1 + U$$

$$\dot{x}_n = -a_{n-1} x_n(t) \dots - a_1 x_2(t) - a_0 x_1(t) + u(t) \quad (a)$$

$$Y(s) = Z(s) . X_1$$

Determination of the output eq^n:

$$Y(s) = b_n s X_n + b_{n-1} X_n + \dots + b_1 X_2 + b_0 X_1$$

$$y(t) = b_n \dot{x}_n(t) + b_{n-1} x_n(t) + \dots + b_1 x_2(t) + b_0 x_1(t)$$

$$= b_n(-a_{n-1} x_n(t) \dots - a_1 x_2(t) - a_0 x_1(t) + u(t)) + b_{n-1} x_n(t) + \dots + b_1 x_2(t) + b_0 x_1(t)$$

$y(t) = (b_{n-1} - b_n a_{n-1}) x_n(t) + \cdots + (b_1 - b_n a_1) x_2(t) + (b_0 - b_n a_0) x_1(t) + b_n u(t)$

The following eqns are available

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_3(t)$$
$$\vdots$$
$$\dot{x}_{n-1}(t) = x_n(t)$$

$$\dot{x}_n(t) = -a_{n-1} x_n(t) - \cdots a_1 x_2(t) - a_0 x_1(t) + u(t)$$
$$y(t) = (b_{n-1} - b_n a_{n-1}) x_n(t) + \cdots + (b_1 - b_n a_1) x_2(t) + (b_0 - b_n a_0) x_1(t) + b_n u(t)$$

The conversion into matrix notation results in the controllability canonical form:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & & \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

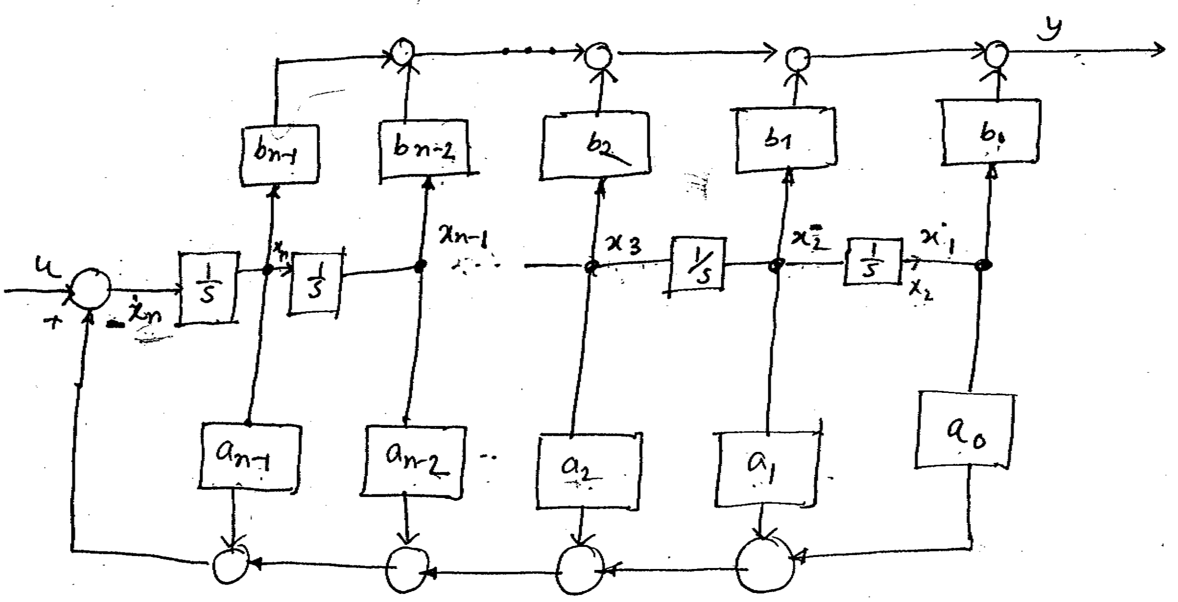
[b → coefficients of numerator]

$$y(t) = [b_0 - b_n a_0, b_1 - b_n a_1 \cdots b_{n-1} - b_n a_{n-1}] \underline{x}(t) + b_n u(t)$$

The controllability canonical form exists only, if the system is controllable.

Flow diagram of a system in the controllability canonical form.

Special case, $b_n = 0$



Obser

Example:

$$G(s) = \frac{s+3}{s^2+4s+1} \qquad n=2$$

$a_0 = 1 \qquad b_0 = 3$

$a_1 = 4 (= a_{n-1}) \quad b_1 = 1 \ (= b_{n-1})$

$b_2 = 0 \ (= b_n)$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(t)$$

$b_0 - b_n a_0$

$= 3 - 0 \times 1$

$b_1 -$

Test:

$\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = -x_1(t) - 4x_2(t) + u(t)$

$\ddot{x}_1(t) = -x_1(t) - 4\dot{x}_1(t) + u(t)$

$\ddot{x}_1(t) + 4\dot{x}_1(t) + x_1 = u(t)$

$X_1(s)(s^2+4s+1) = U(s)$

$y(t) = 3x_1(t) + x_2(t)$

$y(t) = 3x_1(t) + \dot{x}_1(t)$

$Y(s) = (3+s)X_1(s) \longrightarrow Y(s) = \frac{s+3}{s^2+4s+1} U(s)$

$G(s) = \frac{s+3}{s^2+4s+1}$

Observability canonical form:

Linear timeinvariant system of nth order

simplified transfer function in the frequency domain

$$Y(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_n s^{n-1} + \cdots + a_1 s + a_0} U(s) = \frac{Z(s)}{N(s)} U(s)$$

corresponding differential eqn.

$$y^{(n)}(t) + a_{n-1} y^{n-1}(t) + \cdots + a_1 \dot{y} + a_0 y = b_0 u(t) + b_1 \dot{u} + \cdots + b_{n-1} u^{n-1}(t)$$

disadv: derivatives of $u(t)$ on the right-hand side of the eq^n.

Transformation into the frequency domain

$$s^n Y + a_{n-1} s^{n-1} Y + \cdots + a_1 s Y + a_0 Y = b_0 U + b_1 s U + \cdots + b_{n-1} s^{n-1} U$$

$$Y = \frac{1}{s}(b_{n-1} U - a_{n-1} Y + \frac{1}{s}(b_{n-2} U - a_{n-2} Y + \cdots$$

$$+ \frac{1}{s}(b_1 U - a_1 Y +$$

$$+ \frac{1}{s}(b_0 U - a_0 Y)) \cdots))$$

X_1

X_2

X_{n-1}

$X_n = Y$

$$\Downarrow$$

$$y(t) = x_n(t)$$

Substitution: $X_1 = \frac{1}{s}(b_0 U - a_0 X_n)$

$$X_2 = \frac{1}{s}(b_1 U - a_1 X_n + X_1)$$

$$\vdots$$

$$X_n = \frac{1}{s}(b_{n-1} U - a_{n-1} X_n + X_{n-1})$$

$$\Downarrow$$

$$\dot{x}_1(t) = \quad -a_0 x_n(t) + b_0 u(t)$$

$$\dot{x}_2(t) = x_1(t) - a_1 x_n(t) + b_1 u(t)$$

$$\vdots$$

$$\dot{x}_n(t) = x_{n-1}(t) - a_{n-1} x_n(t) + b_{n-1}(t)\, u(t)$$

$$y(t) = x_n(t)$$

Observability canonical form:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & & -a_1 \\ 0 & 1 & & -a_2 \\ \vdots & & & \vdots \\ 0 & 0 & 1 & -a_{n-1} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 0 \quad \cdots \quad 0 \quad 1]\, \underline{x}(t)$$

The observability canonical form exists only, if the system is observable.

Next class diagram of observability

Exercise 1.

a) $\ddot{y}(t) - 6\dot{y}(t) + 9y(t) = 3u(t)$

introduction of state variables

$x_1(t) = y(t) \quad \therefore \dot{y}(t) = \dot{x}_1(t)$

$x_2(t) = \dot{y}(t) \quad \therefore \ddot{y}(t) = \dot{x}_2(t)$

results in $\dot{x}_2(t) - 6\bar{x}_2(t) + 4x_1(t) = 3u(t)$

$\therefore \quad \dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = 6x_2(t) - 4x_1(t) + 3u(t)$

$$\therefore \quad \dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -4 & 6 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u(t)$$

$$\underline{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(t)$$

b) $\ddot{y}(t) + 2y(t) = u(t)$

Let, $y(t) = x_1(t) \therefore \dot{y}(t) = \dot{x}_1(t)$

$\dot{y}(t) = x_2(t)$

$\therefore \ddot{y}(t) = \dot{x}_2(t)$

$\therefore \quad \dot{x}_1(t) = x_2(t) \quad , \quad \dot{x}_2(t) + 2x_1(t) = u(t)$

$\Rightarrow \dot{x}_2(t) = -2x_1(t) + u(t)$

$$\therefore \dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$a_{22} = 0$ because of the absence of $\dot{y}(t)$

$$\underline{y}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}(t)$$

So, missing of 1st derivative $\dot{y}(t)$ results in in the zero element $a_{22} = 0$, in the evolution matrix.

c) $\dddot{y}(t) = u(t)$

Let, $x_1 = y(t) \quad \therefore \dot{y}(t) = \dot{x}_1(t)$

$\dot{y}(t) = x_2(t)$	$\dot{x}_1(t) = x_2(t)$
$\therefore \ddot{y}(t) = \dot{x}_2(t)$	$\dot{x}_2(t) = x_3(t)$
$\ddot{y}(t) = x_3(t)$	$\dot{x}_3(t) = u(t)$
$\therefore \dddot{y}(t) = \dot{x}_3(t)$	$y(t) = x_1(t)$

$$\therefore \dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underline{x}(t)$$

d) $\ddot{y}(t)\dot{y}(t) + 4y(t) = 3u(t)$

The system is nonlinear (two derivatives are multiplied together)
so, we can not do it now.

2) a) The system is
- time invariant
- linear
- single input/output

b) system of 2nd order

c) special case $a_{11} = 0$ $b_1 = 0$, $c_2 = 0$, $d = 0$

$$\dot{x}_1(t) = 3x_2(t)$$
$$\dot{x}_2(t) = 4x_1(t) - 6x_2(t) + b_2 u(t)$$
$$y(t) = c_1 x_1(t)$$

Conversion of the second order eqn:

with, $\dot{x}_1(t) = 3x_2(t) \rightarrow x_2(t) = \frac{1}{3}\dot{x}_1(t)$

$\ddot{x}_1(t) = 3\dot{x}_2(t) \rightarrow \dot{x}_2(t) = \frac{1}{3}\ddot{x}_1(t)$

$$\frac{1}{3}\ddot{x}_1(t) = 4x_1(t) - 6 \times \frac{1}{3}\dot{x}_1(t) + b_2 u(t)$$

$$\Rightarrow \ddot{x}_1(t) = 12x_1(t) - 6\dot{x}_1(t) + 3b_2 u(t) \Rightarrow \ddot{x}_1(t) + 6\dot{x}_1(t) - 12x_1(t) = 3b_2 u(t)$$

conversion of output eqns:

$y(t) = c_1 x_1(t) \rightarrow x_1 = \frac{1}{c_1} y(t)$

$\therefore \dot{x}_1 = \frac{1}{c_1}\dot{y}(t)$

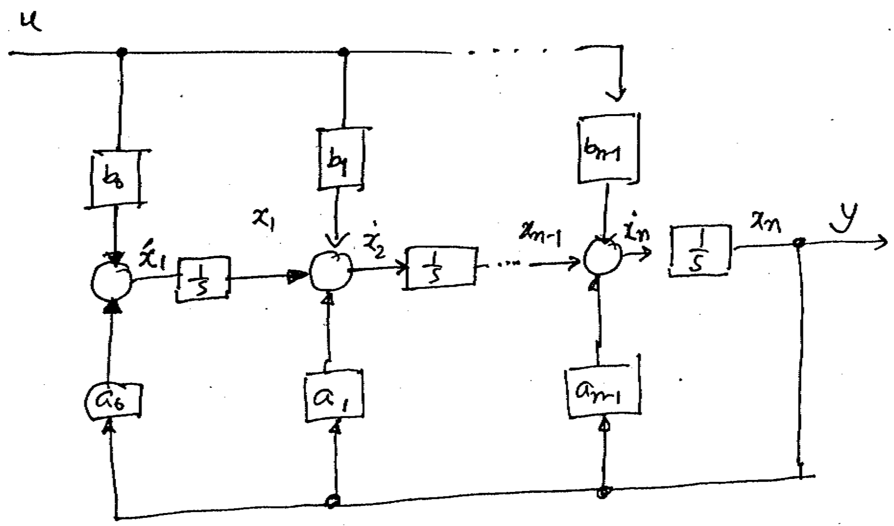
$\ddot{x}_1(t) = \frac{1}{c_1}\ddot{y}(t)$

$$\frac{1}{c_1}\ddot{y}(t) + 6 \times \frac{1}{c_1}\dot{y}(t) - 12 \times \frac{1}{c_1} y(t) = 3b_2 u(t)$$

$$\therefore \ddot{y}(t) + 6\dot{y}(t) - 12y(t) = 3b_2 c_1 u(t).$$

d) The equations are no more solveable in direct way as long as $a_{11} \neq 0$.

flow diagram of a system in the observability canonical form: ($b_n = 0$)



Example: Transfer function

$$G(s) = \frac{s+3}{s^2+4s+1}$$

$a_0 = 1$ $\qquad$ $b_0 = 3$

$a_1 = 4 \ (= a_{n-1})$ $\qquad$ $b_1 = 1 \ (= b_{n-1})$

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}(t)$$

Test: $y(t) = x_2(t)$

$$\frac{d}{dt} : \dot{y}(t) = \dot{x}_2(t)$$

$$y(t) = x_1(t) - 4x_2(t) + u(t)$$

$$\frac{d}{dt} : \ddot{y}(t) = \dot{x}_1(t) - 4\dot{x}_2(t) + \dot{u}(t)$$

$$\dot{y}(t) = -x_2(t) + 3u(t) - 4\dot{x}_2(t) + \dot{u}(t)$$

$$\ddot{y}(t) = -y(t) - 4\dot{y}(t) + 3u(t) + \dot{u}(t)$$

$$\Rightarrow \ddot{y}(t) + 4\dot{y}(t) + y(t) = 3u(t) + \dot{u}(t)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+4s+1}$$

The Jordan Canonical Form:

Controllability & Observability canonical forms:
→ derivation from the co-efficients of the numerator as well as the denominator polynomial of the complex transfer function.

Jordan canonical form:
Derivation from the poles $\lambda_i = s_i$ of the transfer function. All poles must be known.

Development of the Jordan canonical form:

starting point: Complex transfer function

$$Y(s) = \frac{b_n s^n + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} U(s)$$

By decomposition into partial fractions one receives

$$Y(s) = \left(\sum_{i=1}^{n} \frac{c_i}{s - \lambda_i} + d \right) U(s)$$

Definition of the state variables:

$$X_i(s) = \frac{1}{s - \lambda_i} U(s)$$

It results:

$$Y(s) = \sum_{i=1}^{n} c_i X_i(s) + d\, U(s)$$

transformation into the time domain

$$\dot{x}_i(t) = \lambda_i x_i(t) + u(t)$$

$$y(t) = \sum_{i=1}^{n} c_i x_i(t) + d u(t)$$

Jordan canonical form:

$$\dot{\underline{x}}(t) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & 0 \cdots & 0 \\ 0 & \cdots & & \lambda_n \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \underline{x}(t) + d u(t)$$

Book

$\boxed{\text{Franklin}}$

Advantage: Diagonal matrix
- isolated system of differential equations
- Each differential equation can be solved by itself
- useful for pure investigation

Döble poles:

Multi (m-fold) pole at λ_1

it results from the decomposition into parallel fractions :

$$X_1(s) = \frac{1}{s-\lambda_1} U(s)$$
$$X_2(s) = \frac{1}{(s-\lambda_1)^2} U(s)$$
$$\vdots$$
$$X_m(s) = \frac{1}{(s-\lambda_1)^m} U(s)$$
$$X_{m+1}(s) = \frac{1}{s-\lambda_{m+1}} U(s)$$
$$X_n(s) = \frac{1}{s-\lambda_n} U(s)$$

$$x_1(s) = \frac{1}{s-\lambda_1} U(s)$$
$$X_2(s) = \frac{1}{s-\lambda_1} X_1(s)$$
$$X_m(s) = \frac{1}{s-\lambda_1} X_{m-1}(s)$$
$$X_{m+1}(s) = \frac{1}{s-\lambda_{m+1}} U(s)$$
$$X_n(s) = \frac{1}{s-\lambda_n} U(s)$$

inverse laplace transformation

$$\dot{x}_1(t) = \lambda_1 x_1(t) \qquad + u(t)$$
$$\dot{x}_2(t) = \lambda_1 x_2(t) + x_1(t)$$
$$\dot{x}_3(t) = \lambda_1 x_3(t) + x_2(t)$$
$$\vdots$$
$$\dot{x}_m(t) = \cancel{\lambda_{m+}}\ \lambda_1 x_m(t) + x_{m-1}(t)$$
$$\dot{x}_{m+1}(t) = \lambda_{m+1} x_{m+1} + u(t)$$

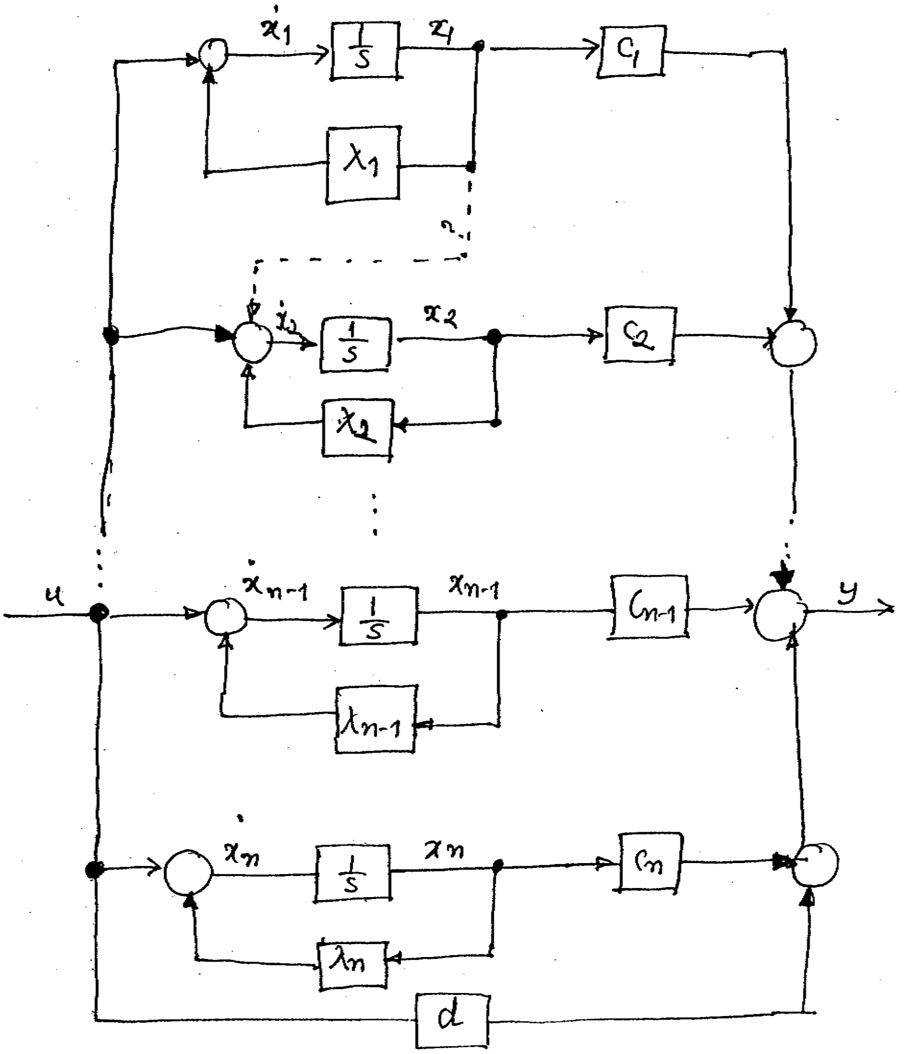
$$\dot{x}_n(t) = \lambda_n x_n(t) \quad + u(t)$$

Jordan canonical from with multipoles:

$$\dot{\underline{x}}(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 & 0 \\ 0 & 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{m+1} & \vdots \\ \vdots & \vdots & \vdots & & \ddots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \underline{x}(t) + d\,u(t)$$

Flow diagram of a system in Jordan Canonical from:



Conjugate complex pole pairs

Given is:

$\dot{x}_1(t) = \lambda_1 x_1(t) + u(t)$ with $\lambda_1 = a + jb$

$\dot{x}_2(t) = \lambda_2 x_2(t) + u(t)$ $\quad\quad \lambda_2 = a - jb$

Approach for conjugated complex state variables:

$$\begin{cases} x_1(t) = \xi(t) + j\eta(t) \\ x_2(t) = \xi(t) + j\eta(t) \end{cases}$$

$$\dot{\xi}(t) + j\dot{\eta}(t) = (a+jb)\left(\xi(t) + j\eta(t)\right) + u(t)$$

$$\dot{\xi}(t) + j\dot{\eta}(t) = a\xi(t) - b\eta(t) + j\left(b\xi(t) + a\eta t)\right) + u(t)$$

The comparison of the real and imaginary part provides

$$\dot{\xi}(t) = a\xi(t) + b\eta(t) + u(t)$$

$$\dot{\eta}(t) = b\xi(t) + a\eta(t)$$

Thus the conjugated complex pair of poles was transferred into a real representation regarding Jordan canonical form.

Disadvantages:

- Transformed state variables represent the system dynamics
- The differential equations of first order are no more decoupled
- In case of existence of further poles as well as ascertaining of the input and the output vector further transformations must be applied.

Consequence:

In the case of existence of conjugated complex pairs of poles the Jordan Canonical form should not be used if possible.

Eigen values & Eigen vectors

The eigen values of a linear system are identical to the poles of the corresponding complex transfer function.

Calculation of the Eigenvalues of the evolution matrix:

starting point: The complex transfer function for a single input/output system

$$G(s) = \underline{c}^T (s\underline{I} - \underline{A})^{-1} \underline{b} + d$$

The poles of the system are determined by the denominator $N(s)$ of the polynomial $G(s)$

$$N(s) = 0$$

The denominator $N(s)$ results exclusively from the matrix inversion of $(s\underline{I} - \underline{A})^{-1}$. consequently the following is valid.

$$\det(s\underline{I} - \underline{A}) = 0$$

The breakup by s_i, $i = 1 .. n$ yields the Eigenvalues λ_i

Definition of the Eigenvalues:

$$\det(\lambda_i \underline{I} - \underline{A}) = 0$$

Characteristic equation of the matrix $\underline{A}$

Eigen vectors:

To each eigenvalue λ_i belongs to an eigenvector $\underline{v}_i$, which is different from zero vector. It fulfils the vector equation

$$(\lambda_i \underline{I} - \underline{A}) \underline{v}_i = \underline{0}$$

The eigen vectors are linear independent from each other.

Example:

starting point is the evolution matrix

$$\underline{A} = \begin{bmatrix} -3 & 3 \\ 1 & -5 \end{bmatrix}$$

$$\det(\lambda_i \underline{I} - \underline{A}) = \det \begin{bmatrix} \lambda_i + 3 & -3 \\ -1 & \lambda_i + 5 \end{bmatrix}$$

$$= (\lambda_i + 3)(\lambda_i + 5) - 3$$

$$= \lambda_i^2 + 8\lambda_i + 15 - 3$$

$$= \lambda_i^2 + 8\lambda_i + 12 = (\lambda_i + 2)(\lambda_i + 6)$$

The characteristic equation yields,

$$(\lambda_i + 2)(\lambda_i + 6) = 0$$

$\rightarrow$ Eigen values $\lambda_1 = -2$, $\lambda_2 = -6$

Eigenvectors:

$$(\lambda_i \underline{I} - \underline{A})\,\underline{v_i} = \underline{0}$$

$$\begin{bmatrix} \lambda_i + 3 & -3 \\ -1 & \lambda_i + 5 \end{bmatrix} \begin{bmatrix} v_{1i} \\ v_{2i} \end{bmatrix} = \underline{0}$$

results in

$$(\lambda_i + 3)\,v_{1i} - 3\,v_{2i} = 0$$

$$-v_{1i} + (\lambda_i + 5)\,v_{2i} = 0$$

one receives for:

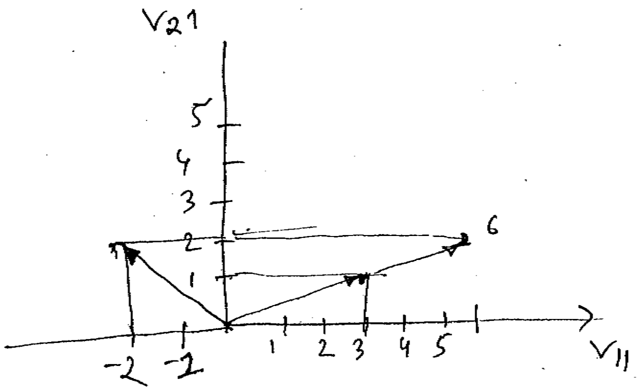
$i = 1$; $\lambda_1 = -2$;

$$\left.\begin{aligned} v_{11} + 3v_{21} &= 0 \\ -v_{11} + 3v_{21} &= 0 \end{aligned}\right\} \quad v_{11} = 3v_{21}$$

$i = 2$; $\lambda_2 = -\cancel{6}$:

$$\left.\begin{aligned} -3v_{12} - 3v_{22} &= 0 \\ -v_{12} - v_{22} &= 0 \end{aligned}\right\} \quad v_{12} = -v_{22}$$

only the direction is important for Eigen vector, not values.



$V_{11} = 3V_{21}$

$\downarrow$

$\frac{1}{}$

for

2

v_{21} & v_{22} can be freely chosen.

$v_{21} = 1$ & $v_{22} = 2$ results in

$$\underline{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \& \quad \underline{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$\underline{v}_1$ & $\underline{v}_2$ are linear independent from each other.

The Equivalent System Transformation

System in the state space Description:

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t)$$

The new state vector: (first define new state vector)

$$\bar{\underline{x}}(t) = \underline{T}\,\underline{x}(t)$$

$\underline{T}$ is non singular $n \times n$ matrix consisting of constant elements,

Besides, it applies

$$\underline{x}(t) = \underline{T}^{-1}\bar{\underline{x}}(t)$$
$$\Rightarrow \dot{\underline{x}}(t) = \underline{T}^{-1}\dot{\bar{\underline{x}}}(t)$$

By inserting into the state equations one receives

$$\underline{T}^{-1}\dot{\bar{\underline{x}}}(t) = \underline{A}\,\underline{T}^{-1}\bar{\underline{x}}(t) + \underline{B}\,\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\,\underline{T}^{-1}\bar{\underline{x}}(t) + \underline{D}\,\underline{u}(t)$$

Multiplication of the state differential equation by $\underline{T}$ from the left shows:

$$\dot{\bar{\underline{x}}}(t) = \underbrace{\underline{T}\,\underline{A}\,\underline{T}^{-1}}_{\bar{\underline{A}}}\bar{\underline{x}}(t) + \underbrace{\underline{T}\,\underline{B}}_{\bar{\underline{B}}}\,\underline{u}(t)$$
$$\underline{y}(t) = \underbrace{\underline{C}\,\underline{T}^{-1}}_{\bar{\underline{C}}}\bar{\underline{x}}(t) + \underbrace{\underline{D}}_{\bar{\underline{D}}}\,\underline{u}(t)$$

Hereof the transformation equations results in

$$\boxed{\begin{aligned} \bar{\underline{A}} &= \underline{T}\,\underline{A}\,\underline{T}^{-1} \\ \bar{\underline{B}} &= \underline{T}\,\underline{B} \\ \bar{\underline{C}} &= \underline{C}\,\underline{T}^{-1} \\ \bar{\underline{D}} &= \underline{D} \end{aligned}}$$

The transformed state equations are

$$\dot{\bar{x}}(t) = \bar{A}\,\bar{x}(t) + \bar{B}\,u(t)$$
$$y(t) = \bar{C}\,\bar{x}(t) + D\,u(t)$$

Eigenvectors dont belong to complex transfer function

So, Eigen values of these eqns are not same as before.
? vectors ?

The initial values for the transformed system result in

$$\bar{x}(t_0) = T\,x(t_0)$$

Input/Output behavior of the transformed system:

$$\bar{G}(s) = \bar{C}\,(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$
$$= C\,T^{-1}(sI - T A T^{-1})^{-1} T\, B + D$$

[double applying the rule for matrix calculation:
$F^{-1} \cdot E^{-1} = (E F)^{-1}$

$$\bar{G}(s) = C\,(s T^{-1} I T - T^{-1} T A T^{-1} T)^{-1} B + D$$
$$= C\,(sI - A)^{-1} B + D$$
$$\rightarrow \bar{G}(s) = G(s)$$

Conclusion:

The equivalent transformation causes

* a modified representation of system-internal dynamic process caused by the transformed state vector $\bar{x}(t)$
* no modification regarding the dynamic input/output behavior of the transformed system compared to the original system, which was never transformed.

With the equivalent system transformation ~~of~~ the state vector $x(t)$ is transformed. The input vector $u(t)$ as well as the output vector $y(t)$ remain unmodified.

The transformation of a system into the Jordan Canonical form:

System in the state space description

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$

$$\underline{y}(t) = \underline{C}\,x(t) + \underline{D}\,\underline{u}(t)$$

Transformation

$$\underline{x}(t) = \underline{V}\,\underline{\bar{x}}(t) \quad \text{with} \quad \underline{V} = T^{-1}$$

The transformation matrix $\underline{V}$ can be indicated in coulumn vector notation as follows,

$$\underline{V} = [\underline{v}_1, \underline{v}_2, \dots \underline{v}_n] \quad \text{with} \quad \underline{v}_i = \begin{bmatrix} v_{1i} \\ v_{2i} \\ \vdots \\ v_{ni} \end{bmatrix} \qquad i = 1 \dots n$$

The system transformation into the Jordan Canonical form results in the following approach:

$$\dot{\bar{\underline{x}}}(t) = \underbrace{\underline{V}^{-1}\,\underline{A}\,\underline{V}(t)}_{\boldsymbol{\Lambda}}\,\bar{\underline{x}} + \underline{V}^{-1}\,\underline{B}\,\underline{u}$$

Thus it is valid :

$$\underline{V}^{-1}\,\underline{A}\,\underline{V} = \boldsymbol{\Lambda} \quad \text{with} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\underline{A}\,\underline{V} = \underline{V}\boldsymbol{\Lambda}$$

left side :

$$\underline{A}\,\underline{V} = \underline{A}\,[\underline{v}_1 \cdots \underline{v}_n]$$

$$= [\underline{A}\,\underline{v}_1 \cdots \underline{A}\,\underline{v}_n]$$

right side :

$$\underline{V}\,\boldsymbol{\Lambda} = \underline{V}\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} v_{11}\lambda_1 & v_{12}\lambda_2 & \cdots & v_{1n}\lambda_n \\ \vdots & & & \vdots \\ v_{n1}\lambda_1 & v_{n2}\lambda_2 & \cdots & v_{nn}\lambda_n \end{bmatrix}$$

$$\underline{v}_1\lambda_1 \quad \underline{v}_2\lambda_2 \quad \underline{v}_n\lambda_n$$

$$= [\lambda_1\underline{v}_1 \cdots \lambda_n\underline{v}_n]$$

$$\Rightarrow \left[\underline{A}\,\underline{v}_1 \cdots \underline{A}\,\underline{v}_n\right] = \left[\lambda_1 \underline{v}_1 \cdots \lambda_n \underline{v}_n\right]$$

It applies to every column:

$$\underline{A}\,\underline{v}_1 = \lambda_1 \underline{v}_1$$

$$\Rightarrow (\lambda_1 \underline{I} - \underline{A})\underline{v}_1 = 0$$

The operation $(\lambda_i \underline{I} - \underline{A})\,\underline{v}_i = \underline{0}$ corresponds exactly to the determining equation of the Eigen vectors.

Consequently one receives

$$\underline{V} = \left[\underline{v}_1,\ \underline{v}_2 \cdots \underline{v}_n\right]$$

The transformation matrix $\underline{V}$ is composed columnwise from the Eigen vectors $\underline{v}_i$ corresponding to the ~~Eigenvectors λ_i~~, Eigen values λ_i.

Practical Procedure:

starting point:

$$\dot{\underline{x}}(t) = \underline{A}\,\cancel{(x)}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$

$$y(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t)$$

· Step 1: Find the Eigenvalues from the characteristic equation

$$\det(\lambda_i \underline{I} - \underline{A}) = 0$$

· Step 2: Find the Eigenvectors $\underline{v}_i$, which belong to Eigen values λ_i respectively, according to

$$(\lambda_i \underline{I} - \underline{A})\,\underline{v}_i = 0$$

Here one element of $\underline{v}_i$, ~~which belong to eigenvalues λ_i~~ can be chosen normally freely. Then, the remaining elements result from the element.

· Step 3 put together the "the transformation matrix" $\underline{V}$.

$$\underline{V} = \left[\underline{v}_1,\ \underline{v}_2 \cdots \underline{v}_n\right]$$

· Step 4

Calculate the matrices:

$$\underline{\Lambda} = \underline{V}^{-1}\,\underline{A}\,\underline{V}$$
$$\underline{\bar{B}} = \underline{V}^{-1}\underline{B}$$
$$\underline{\bar{C}} = \underline{C}\,\underline{V}$$
$$\underline{\bar{D}} = \underline{D}$$

· step 5':
set up the equations in the Jordan Canonical form (first general form)

$$\dot{\bar{\underline{x}}}(t) = \underline{\Lambda}\,\bar{\underline{x}}(t) + \underline{\bar{B}}\,\underline{u}(t)$$
$$\underline{y}(t) = \underline{\bar{C}}\,\bar{\underline{x}}(t) + \underline{\bar{D}}\,\underline{u}(t)$$

with $\underline{\Lambda} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$ → By further standardization at last the Jordan Canonical form come up.

Influence of the system Transformation on the controllability and observability:

Controllability matrix for the transformed system:

$$\bar{Q}_c = [\underline{\bar{B}},\ \underline{\bar{A}}\,\underline{\bar{B}} \cdots \underline{A}^{n-1}\underline{B}]$$

It applies to $i = 0, 1, \cdots n-1$

$$\underline{\bar{A}}\,\underline{\bar{B}} = \underbrace{\underline{T}\,\underline{A}\,\underline{T}^{-1}}_{\bar{A}}\,\underbrace{\underline{T}\,\underline{A}\,\underline{T}^{-1}}_{\bar{A}} \cdots \underbrace{\underline{T}\,\underline{A}\,\underline{T}^{-1}}_{\bar{A}}\,\underbrace{\underline{T}\,\underline{B}}_{\bar{B}}$$

$$= \underline{T}\,\underline{A}^{i}\,\underline{B} \quad ? \ \mathrm{T\,A^{i}\,B}$$

$$\bar{Q}_c = [\underline{T}\,\underline{B},\ \underline{T}\,\underline{A}\,\underline{B} \ \cdots \ \underline{T}\,\underline{A}^{n-1}\underline{B}]$$
$$= \underline{T}\,[\underline{B},\ \underline{A}\,\underline{B} \cdots \underline{A}^{n-1}\underline{B}]$$

$$\therefore \bar{Q}_c = \underline{T}\,Q_c$$

The corresponding applies to observability matrix

$$\bar{Q}_o = \begin{bmatrix} \underline{\bar{C}} \\ \underline{\bar{C}}\,\underline{\bar{A}} \\ \vdots \\ \underline{\bar{C}}\,\underline{\bar{A}}^{n-1} \end{bmatrix} = \begin{bmatrix} \underline{C}\,\underline{T}^{-1} \\ \underline{C}\,\underline{A}\,\underline{T}^{-1} \\ \vdots \\ \underline{C}\,\underline{A}^{n-1}\,\underline{T}^{-1} \end{bmatrix} = \begin{bmatrix} \underline{C} \\ \underline{C}\,\underline{A} \\ \vdots \\ \underline{C}\,\underline{A}^{n-1} \end{bmatrix}\underline{T}^{-1}$$

$$\therefore \bar{Q}_o = \underline{Q}_o\,\underline{T}^{-1}$$

As $\underline{T}$ is not singular, the rank of a matrix will not be altered during a multiplication with $\underline{T}$

→ controllability & observability of a linear timeinvariant is transformation invariant.

Control with complete state feed back

starting point:

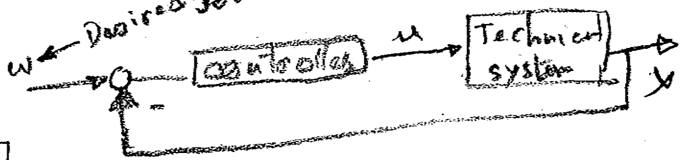
Time invariant, linear, controllable, single input/output system:

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{b}\,u(t)$$

$$y(t) = \underline{c}^T\,\underline{x}(t)$$

As usual in practice, ~~in~~ the value of $\underline{d}$ might be set to zero here.

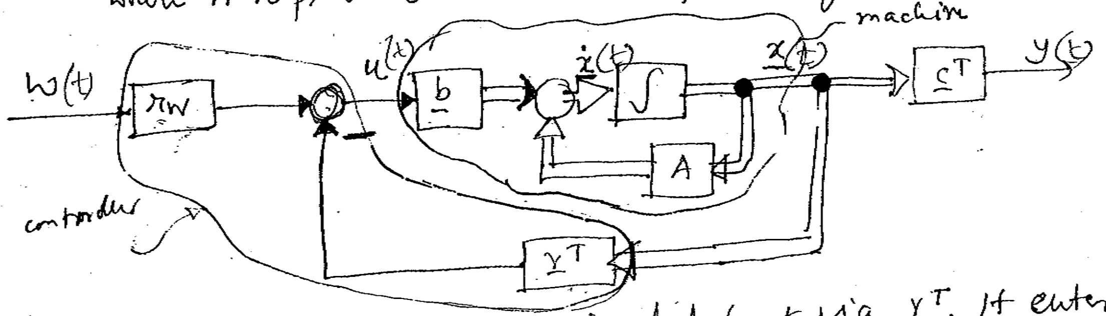
Feedback of the state vector



$$u(t) = r_w\,w(t) - \underline{r}^T\underline{x}(t)$$

with $\underline{r}^T = [r_1, r_2 \ldots r_n]$

where n represents the order of the system



The complete state vector $\underline{x}(t)$ is fed back via $\underline{r}^T$. It enters the control algorithm.

Behavior of the overall system ?

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{b}\,u(t)$$

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{b}\left(r_w\,w(t) - \underline{r}^T\underline{x}(t)\right)$$

v.v.g. ***

$$\boxed{\underline{\dot{x}}(t) = (\underline{A} - \underline{b}\,\underline{r}^T)\,\underline{x}(t) + \underline{b}\,r_w\,w(t)} \quad \boxed{\text{overall eq}^{n}}$$

$$s\,\underline{X}(s) = (\underline{A} - \underline{b}\,\underline{r}^T)\,\underline{X}(s) + \underline{b}\,r_w\,W(s)$$

$$\Rightarrow (s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T)\,\underline{X}(s) = \underline{b}\,r_w\,W(s)$$

$\Rightarrow \underline{X}(s) = (s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T)^{-1}\,\underline{b}\, r_w\, W(s)$

The poles of the transfer polynomial results from the characteristic equation :

$$\det(s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T) = 0$$

Equivalent approch for the characteristic eqn

$$(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n) = 0$$

Comparing these two approaches yields :

$$\det(s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$

Procedure for the concrete application here :

- step 1 : Calculate the left side of the equation by inserting the elements for $\underline{A}$ & $\underline{b}$

  The demanded values for $\underline{r}^T = [r_1, r_2 \ldots r_n]$ still remain in general form

- step 2 : Calculate the right hand side of the equation by selection of the eigen values (Poles) $\lambda_1, \lambda_2 \ldots \lambda_n$ for the overall system

- step 3 : perform comparison of the co-efficients for both sides of the equation with regard to all powers of the complex frequency s.

- step 4 : Find the solution of the n equations which n unknowns

  Now the vector $\underline{r}^T$ is found

Prerequisite for the derivation of a state feedback controller is the Controllability of the system

—x—

$[r_1, r_2 \cdots r_n]$

$\underline{r}^T \to$ row vector

$r_w \to$ scalar value

Example: $\dot{x}(t) = \begin{bmatrix} -3 & 1 \\ 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$

$y(t) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} x(t)$

r_N, r^T

Assertion of the controllability:

$$\underline{Q}_c = [\underline{b} \quad \underline{A}\,\underline{b}] = \begin{bmatrix} 1 & -1 \\ 2 & -8 \end{bmatrix} \rightarrow \text{rank } \underline{Q}_c = 2 = n$$

→ the system is controllable,
a state feed back controller can be developed

problem: The Eigen values of the closed loop should be located at
$\lambda_1 = -1$, $\lambda_2 = -2$ [because, these roots make the system stable]

$$\det(s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T) = (s-\lambda_1)(s-\lambda_2)$$

1. Left side of the equation with $r^T = [r_1 \quad r_2]$

$$\det(s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T) = \det \begin{bmatrix} s+3+r_1 & -1+r_2 \\ 2r_1 & s+4+2r_2 \end{bmatrix}$$

$$= s^2 + (r_1+2r_2+7)s + 6r_1 + 6r_2 + 12$$

2. Right hand side of the equation

$$(s-\lambda_1)(s-\lambda_2) = (s+1)(s+2)$$
$$= s^2+3s+2$$

3. comparison of eq^{n}s:

$r_1 + 2r_2 + 7 = 3$ — ①

$6r_1 + 6r_2 + 12 = 2$ — ②

s: $r_1 = -4 - 2r_2$

s^0:

4. solution:

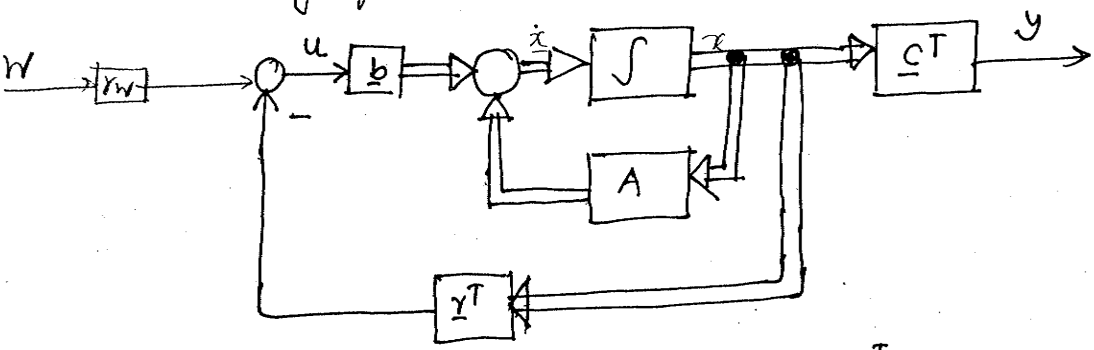
$-24 - 12r_2 + 6r_2 + 12 = 2$

$6r_2 = -14 \rightarrow r_2 = -\frac{14}{6} = -\frac{7}{3}$

$r_1 = -4 + 2\cdot\frac{7}{3} = \frac{2}{3}$

$\therefore \underline{r}^T = \begin{bmatrix} \frac{2}{3} & -\frac{7}{3} \end{bmatrix}$

The following system structure is now available:



The feedback branch is implemented by $\underline{r}^T$
However, the nominal branch is not considered yet

· Ascertaintment of the gain of the desired value

Output equation:

$$y(t) = \underline{c}^T \underline{x}(t)$$

$$Y(s) = \underline{c}^T \underline{X}(s)$$

with $\underline{X}(s) = (s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T)^{-1}\,\underline{b}\, r_W\, W(s)$
results in the complex transfer function,

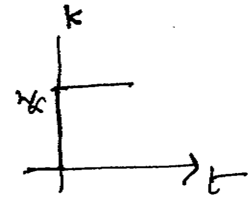
$$Y(s) = \underbrace{\underline{c}^T (s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T)^{-1}\,\underline{b}\, r_W}_{F_W(s)}\, W(s)$$

It is required that in the steady state $y_\infty = w_\infty$ should be valid,
ie the controlled value must adapt itself to the nominal value.
Considering this, r_w must be callibrated

The following is valid:

$$Y(s) = F_w(s) \cdot W(s)$$

Final value theorem of the Laplace transform



$$y_\infty = \lim_{s\to 0} s \cdot F_w(s) \cdot W(s) \quad \text{with } W(s) = \frac{1}{s} w_\infty$$

?

$$= \lim_{s\to 0} F_w(s) \cdot s \cdot \frac{1}{s} w_\infty$$

$$y_\infty = \lim_{s\to 0} F_w(s) \cdot w_\infty$$

with, $F_w(s) = \underline{c}^T (s\underline{I} - \underline{A} + \underline{b}\,\underline{r}^T)^{-1}\,\underline{b}\, r_w$

$$\therefore\ y_\infty = \underline{c}^T (\underline{b}\,\underline{r}^T - \underline{A})^{-1}\,\underline{b}\, r_w\, w_\infty$$

with $y_\infty = w_\infty$

$$\underline{c}^T (\underline{b}\,\underline{r}^T - \underline{A})^{-1}\,\underline{b}\, r_w = 1$$

→ Equation for the calculation of the gain for the nominal value r_w

$$r_w = \left(\underline{c}^T(\underline{b}\,\underline{r}^T - \underline{A})^{-1}\underline{b}\right)^{-1}$$

Supplement to the procedure listed above for the concrete application:

• step 5: Calculate the required gain r_w for the nominal value in accordance to the equation shown above.

Annotation:

The equation is based on the fact that in the steady-state case the value for the output y_∞ is equal to the nominal value w_∞.

Continuation of the example:

5. Calculation of r_w

$$\underline{b}\,\underline{r}^T - \underline{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{7}{3} \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 6 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}+3 & -\frac{7}{3}-1 \\ 4/3 & -\frac{14}{3}+4 \end{bmatrix} = \begin{bmatrix} \frac{11}{3} & -\frac{10}{3} \\ \frac{4}{3} & -\frac{2}{3} \end{bmatrix}$$

$$(\underline{b}\,\underline{r}^T - \underline{A})^{-1} = \frac{\begin{bmatrix} -\frac{2}{3} & \frac{10}{3} \\ -4/3 & \frac{11}{3} \end{bmatrix}}{-\frac{22}{9} + \frac{40}{9}} = \begin{bmatrix} -\frac{1}{3} & \frac{5}{3} \\ -\frac{2}{3} & \frac{11}{6} \end{bmatrix}$$

$$(\underline{b}\,\underline{r}^T - \underline{A})^{-1}\underline{b} = \begin{bmatrix} -\frac{1}{3} & \frac{5}{3} \\ -\frac{2}{3} & \frac{11}{6} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\underline{c}^T(\underline{b}\,\underline{r}^T - \underline{A})^{-1}\underline{b} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3$$

$$\left(\underline{c}^T(\underline{b}\,\underline{r}^T - \underline{A})^{-1}\underline{b}\right)^{-1} = \frac{1}{3}$$

→ r_w (Input gain) $= \frac{1}{3}$

for better understanding:

$$y_\infty = \underbrace{\underline{c}^T(\underline{b}\,\underline{r}^T - \underline{A})^{-1}\underline{b}}_{3} \cdot \underbrace{r_w}_{1/3} \cdot w_\infty$$

→ $y_\infty = w_\infty$

Special case: The system exists already in the controllability canonical form

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & & & 1 \\ -a_0 & -a_1 & -a_2 & & -a_{n-1} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

These are from transfer functions

$$y(t) = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} \underline{x}(t)$$

in general first is to consider: with $b_n = 0$

introduction of a state feedback controller

$$u(t) = r_w w(t) - \underline{r}^T \underline{x}(t) \quad \text{with } \underline{r}^T = [r_1\ r_2 \cdots r_n]$$

State differential equation:

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{b}\,u(t) \rightarrow \text{(for the technical system)}$$

controllability canonical matrix. Co

Control matrix $B = 0 \ldots$

overall system:

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{b}\left(r_w w(t) - \underline{r}^T \underline{x}(t)\right)$$

$$\dot{\underline{x}}(t) = \underbrace{(\underline{A} - \underline{b}\,\underline{r}^T)}_{\underline{A}_R} \underline{x}(t) + \underline{b}\, r_w\, w(t) \quad \cdots\cdots \text{ⓐ}$$

With the special case of the controllability canonical form the evolution matrix $\underline{A}_R$ of the overall system results in:

$$\underline{A}_R = \underline{A} - \underline{b}\,\underline{r}^T$$

$$= \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ -a_0 & -a_1 & -a_2 & & -a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} [r_1, r_2, \cdots r_n]$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & 1 \\ \underbrace{-(a_0 + r_1)}_{a_{R,0}} & \underbrace{-(a_1 + r_2)}_{a_{R,1}} & \vdots & & \underbrace{-(a_{n-1} + r_n)}_{a_{R,n-1}} \end{bmatrix}$$

The following is valid,

$$a_{R,0} = a_0 + r_1$$

$$a_{R,1} = a_1 + r_2$$

$$\vdots$$

$$a_{R,n-1} = a_{n-1} + r_n$$

$a_{R,0}$

$= a_0 + r_1$

$a_0 + r_1$

$$r_w = \frac{a_{R,0}}{b_0}$$

from this characteristic polynomial for the overall system can be calculated as follows:

$$\boxed{s^n + a_{R,n-1}s^{n-1} + \dots + a_{R,1}s + a_{R,0} = 0}$$

By a comparison of co-efficients pole placement can be executed

Ascertainment of input gain r_w:

→ considering the steady state case:

Here it should be valid: $y_\infty = w_\infty$ (request !!!)

It applies to the steady state case: $\underline{\dot{x}} = \underline{0}$

from the last line of the differential eqn results,

from ⓐ $\Rightarrow \dot{x}_n = -a_{R,0}\,x_1 - a_{R,1}\,\underset{\uparrow \atop \dot{x}_1}{x_2} - a_{R,2}\,\underset{\uparrow \atop \dot{x}_2}{x_3} - \dots - a_{R,n-1}\,\underset{\uparrow \atop \dot{x}_{n-1}}{x_n} + 1\, r_w w$

~~(also from matrix) state eqn.~~

$\rightsquigarrow 0 = -a_{R,0}\,x_1 + r_w w$

$x_1 = \dfrac{1}{a_{R,0}}\, r_w w$

$y = b_0 x_1 + b_1 \underset{\uparrow \atop \dot{x}_1}{x_2} + b_2 \underset{\uparrow \atop \dot{x}_2}{x_3} + \dots\, b_{n-1} \underset{\uparrow \atop x_{n-1}}{x_n}$

$\rightsquigarrow y = b_0 x_1$

$= b_0 \times \dfrac{1}{a_{R,0}}\, r_w w$

on the other hand for $t \to \infty$ it is also valid

$y = w$

$\rightsquigarrow \dfrac{b_0}{a_{R,0}}\, r_w = 1 \quad \rightsquigarrow \boxed{r_w = \dfrac{a_{R,0}}{b_0}}$

$a_{R,n-1} = a_{n-1} + r_n$

<u>next week exercise 2</u>

$s^{n} + a_{R,n-1}\, s^{n-1} + \dots + a_{R,0} = (s-\lambda_1)(s-\lambda_2)\dots(s-\lambda_n)$

s^1

Solutions: dec-9

17/12/2002

Exercise 2

1. Task:

a) $\dot{x}_1(t) = a_{11}x_1(t) + 3x_2(t) + 0$

$\dot{x}_2(t) = 4x_1(t) - 6x_2(t) + b_2 u(t)$

$y(t) = c_1 x_1(t) + c_2 x_2(t) + d\,u(t)$

$\circ\!-\!\bullet$

$sX_1(s) = a_{11}X_1(s) + 3X_2(s)$

$sX_2(s) = 4X_1(s) - 6X_2(s) + b_2 U(s)$

$Y(s) = c_1X_1(s) + c_2X_2(s) + dU(s)$

~~conversion of the eqns in the frequency domain~~; from the 1st eqn follows:

$(s-a_{11})X_1 = 3X_2$

$$X_1 = \frac{3X_2}{s-a_{11}}$$

inserted into the 2nd eqn

$$sX_2 = 4\cdot\frac{3X_2}{s-a_{11}} - 6X_2 + b_2U$$

$$\Rightarrow s^2X_2 - sa_{11}X_2 = 12X_2 - 6sX_2 + 6a_{11}X_2 + b_2sU - b_2a_{11}U$$

$$X_2 = \frac{b_2s - b_2a_{11}}{s^2 + s(6-a_{11}) - (12+6a_{11})}U$$

$$\therefore X_1 = \frac{3\cdot\dfrac{b_2s - b_2a_{11}}{s^2+s(6-a_{11})-(12+6a_{11})}}{s-a_{11}}U$$

$$\Rightarrow X_1 = \frac{3b_2}{s^2+s(6-a_{11})-(12+6a_{11})}U$$

$$Y = c_1X_1 + c_2X_2 + dU$$

$$= \frac{ds^2 + (6d + c_2b_2 - da_{11})s + (3c_1b_2 - c_2b_2a_{11} - 12d - 6a_{11}d)}{s^2 + s(6-a_{11}) - (12+6a_{11})}\cdot U$$

$\bullet\!-\!\circ$

inverse Laplace transformation

$$\ddot{y}(t) + (6-a_{11})\dot{y}(t) - (12+6a_{11})y(t) = (3c_1b_2 - c_2b_2a_{11} - 12d - 6a_{11}d)u(t) + (6d + c_2b_2 - da_{11})\dot{u}(t) + d\ddot{u}(t)$$

(b) with $d=0$ the 2nd derivative of the input variable u disappears.

c) result of a) with $a_{11}=0$, $c_2=0$ and $d=0$

$$\ddot{y}(t)+6\dot{y}(t)-12\,y(t)=3c_1 b_2\, u(t)$$

$\longrightarrow$ Conformity

2) a) Time invariant, linear, multivariable system

b) $\underline{G}(s)=\underline{C}(sI-\underline{A})^{-1}\underline{B}+\underline{D}$

$$G(s)=\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}s+2 & -1\\ 0 & s+1\end{bmatrix}^{-1}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & d_{22}\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}\frac{\begin{bmatrix}s+1 & 1\\ 0 & s+2\end{bmatrix}}{(s+2)(s+1)}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & d_{22}\end{bmatrix}$$

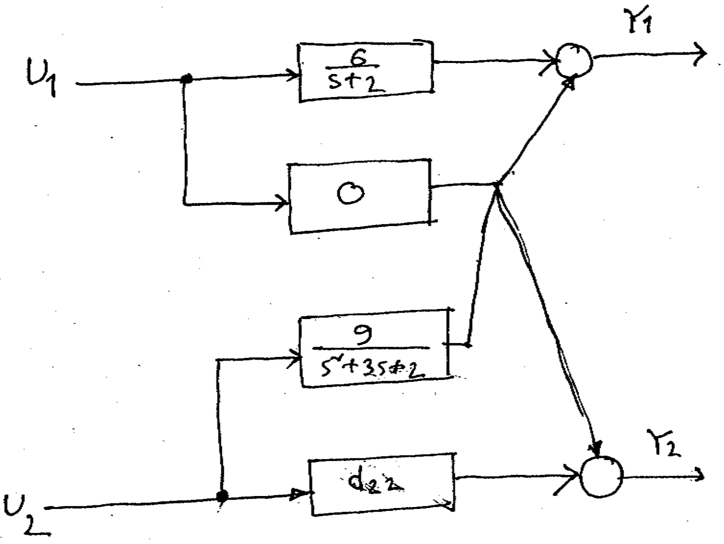
$\vdots$

$$G(s)=\begin{bmatrix}\frac{6}{s+2} & \frac{9}{s^2+3s+2}\\ 0 & d_{22}\end{bmatrix}$$

(c) $\underline{Y}(s)=\underline{G}(s)\cdot\underline{U}(s)$

$$Y_1(s)=\frac{6}{s+2}\,U_1(s)+\frac{9}{s^2+3s+2}\,U_2(s).$$

$$Y_2(s)=\underset{\substack{\downarrow\\ 0\cdot U_1(s)+}}{}d_{22}\,U_2(s)$$



c) Action of Signals:

$u_1(t)\rightarrow y_1(t)$:

effect via the control matrix, evolution matrix, observation matrix

$u_1(t)\longrightarrow y_2(t)$

no effect.

$u_2(t) \rightarrow y_1(t)$

effect via the control matrix

Couplings in the evolution matrix, observation matrix

$u_2(t) \rightarrow y_2(t)$:

Exclusively effect via the ~~to~~ direct transmission matrix

Exercise 3

1. (a) At both systems no couplings are present in each evolution matrix. That means, every state variable must have its own input through the system input (in case of controllability) ~~ass~~ as well as every state variable must be observable (in case of observability) through the system output in case of observability.

first example: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in ~~the~~ control matrix '0' element.

So, not controllable

2nd example: $\begin{bmatrix} 0 & 4 \end{bmatrix}$ observation matrix, element 0

So, not observable.

1st system:

- not controllable (zero element in $\underline{b}$)
- observable (no zero element in $\underline{c}^T$)

2nd system:

- controllable (no zero element in $\underline{b}$)
- not observable (zero element in $\underline{c}^T$)

(b) Controllability matrix

$$\underline{Q}_c = [\underline{b}, \underline{A}\,\underline{b}]$$

$$= \left[\begin{matrix} 1, \\ 2, \end{matrix} \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

rank $\underline{Q}_c = 1 \neq 2$

→ the matrix is not controllable

Observability matrix

$$\underline{Q}_o = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T \underline{A} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ \begin{bmatrix} 2 & 4 \end{bmatrix} & \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \end{bmatrix}$$

$$\Rightarrow \underline{Q}_o = \begin{bmatrix} 2 & 4 \\ -4 & -12 \end{bmatrix}$$

rank $\underline{Q}_o = 2$

→ the system is observable.

2nd System:

Controllability matrix

$$Q_c = [\underline{b}, \underline{A}\,\underline{b}]$$

$$= \begin{bmatrix} 1, & \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ 2, & \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -6 \end{bmatrix}$$

rank $Q_c = 2$

$\rightarrow$ the system is controllable

Observability matrix

$$Q_0 = \begin{bmatrix} c^T \\ c^T A \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ [0 \quad 4] & \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 0 & 4 \\ 0 & -12 \end{bmatrix}$$

rank $Q_0 = 1 \neq 2$

$\rightarrow$ the system is ~~not observable~~

2. Co-efficients,

$a_0 = 0$ $\quad$ $b_0 = 2$

$a_1 = 1$ $\quad$ $b_1 = 0$

$a_2 = 3$ $\quad$ $b_2 = 4$

$(b_n =)\ b_3 = 1$

Controllability Canonical form:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_0 - b_n a_0 \mid b_1 - b_n a_1 \mid b_2 - b_n a_2]\, \underline{x} + b_n u$$

$\therefore$ for the present system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \underline{x}(t) + 1 \cdot u(t) \rightsquigarrow \text{controllable}.$$

$$Q_c = [\underline{b} \quad \underline{A}\,\underline{b}, \quad \underline{A}^2 \underline{b}]$$

b) The system is controllable
If the controllability canonical form exists,
the controllability is principally present.

3. The shown system is neither controllable nor observable. As both state variables $x_1(t)$ and $x_2(t)$ show the same dynamic behavior, they can not be stimulated separately by the system input. Furthermore, taking the equality in the dynamics into account, at the system output no distinction betw the influences of the state variable is possible.

Proof:

$$\underline{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{c}^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Controllability:

$$\underline{Q}_c = \begin{bmatrix} \underline{b} & \underline{A}\,\underline{b} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Rank $\underline{Q}_c = 1 \neq 2 \rightarrow$ not controllable

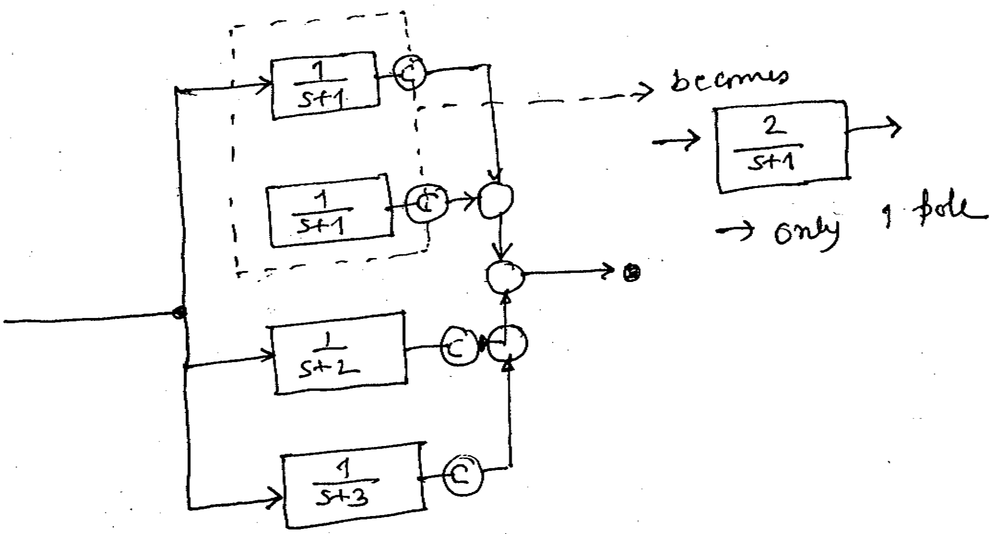
Observability:

$$\underline{Q}_o = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T A \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

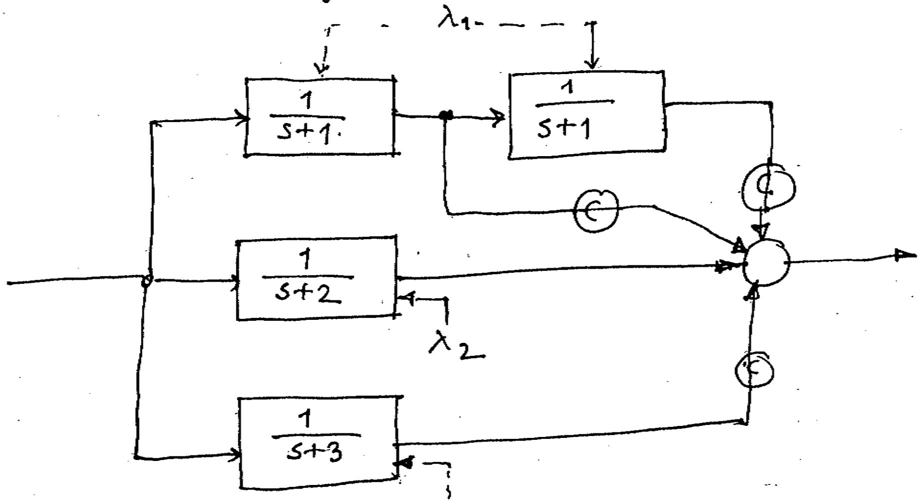
Rank $\underline{Q}_o = 1 \neq 2 \rightarrow$ not observable

Additional considerations regarding the Jordan Canonical form (not component of the solution)

Parallel connection of 2 identical poles yields 1 pole



w.r.t the rules of partial decomposition, the following is valid:



$$\dot{\underline{x}}(t) = \begin{bmatrix} \lambda_1 & & & 0 \\ 1 & \lambda_1 & & \\ & & \lambda_2 & \\ 0 & & & \lambda_3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

the corresponds to the rules of decomposition...

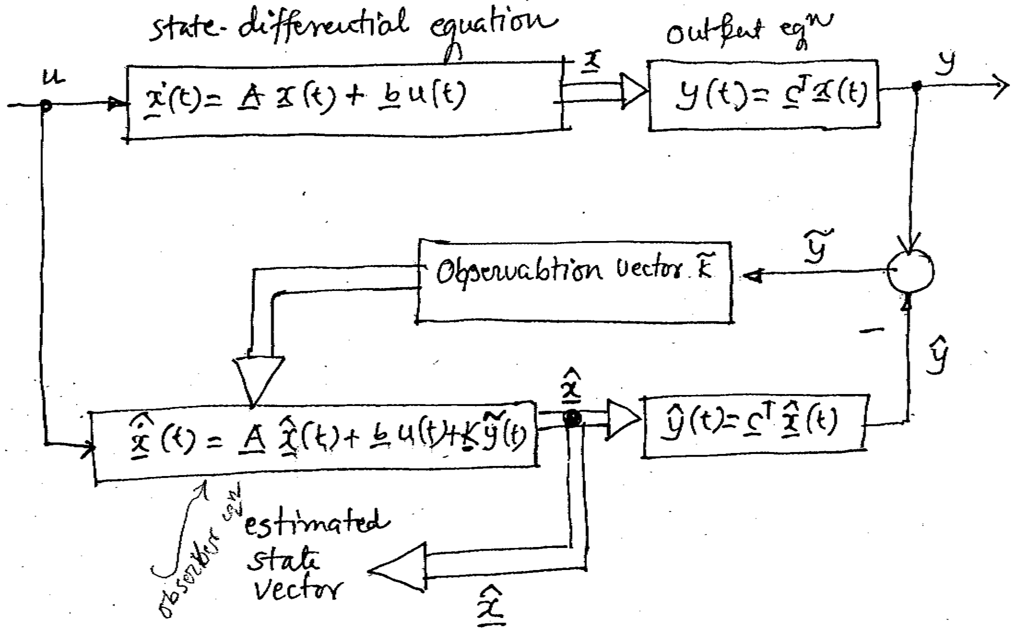
2X2 $\quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

State Estimation by an Observer

Problem:

For a system, whose only the input and the output variables can be measured the prerequisite for the design of a state feedback controller shall be established. Here the system dynamics are well known.

⇝ Computational reconstruction of the progression of the state variables by an observer.



The feedback of the estimation error $\tilde{y}$ between the measured output variable y and the estimated output variable $\hat{y}$ into the observer equations prevents from diverging of $\hat{y}$ from y. This is also valid in case of short term disturbances into the controlled system.

State equations of the system: (single input/output system with $d=0$)

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{b}\,u(t)$$
$$y(t) = \underline{c}^T\underline{x}(t)$$

Resulting from this the observer equations are

$$\dot{\hat{\underline{x}}}(t) = \underline{A}\,\hat{\underline{x}}(t) + \underline{b}\,u(t) + \underline{k}\,(y(t) - \hat{y}(t))$$

$$\hat{y}(t) = \underline{c}^T\hat{\underline{x}}(t)$$

Calculation of the estimation error $\tilde{\underline{x}}(t) = \underline{x}(t) - \hat{\underline{x}}(t)$

$$\begin{aligned}
\dot{\tilde{\underline{x}}}(t) &= \underline{A}\,\underline{x}(t) + \cancel{\underline{b}\,u(t)} \\
&\quad - \underline{A}\,\hat{\underline{x}}(t) - \cancel{\underline{b}\,u(t)} - \underline{k}\left(y(t) - \hat{y}(t)\right) \\
&= \underline{A}\left(\underline{x}(t) - \hat{\underline{x}}(t)\right) - \underline{k}\left(\underline{c}^T\underline{x}(t) - \underline{c}^T\hat{\underline{x}}(t)\right) \\
&= \underline{A}\,\tilde{\underline{x}}(t) - \underline{k}\,\underline{c}^T\tilde{\underline{x}}(t)
\end{aligned}$$

→ Estimation error differential equation

$$\boxed{\dot{\tilde{\underline{x}}}(t) = (\underline{A} - \underline{K}\,\underline{C}^T)\,\tilde{\underline{x}}(t)}$$

→ $\tilde{\underline{A}} = \underline{A} - \underline{K}\,\underline{C}^T \Rightarrow \dot{\tilde{\underline{x}}}(t) = \tilde{\underline{A}}\,\tilde{\underline{x}}(t)$

It is required that with any initial state x_0, $\hat{x}_0$ the estimation error $\tilde{\underline{x}}(t)$ for $t \to \infty$ converges to $\tilde{\underline{x}}(t) \to 0$

This happens precisely in the case, in which the matrix $\tilde{\underline{A}}$ consists exclusively of eigenvalues with negative real parts.

- <u>Calculations of the observation vector $\underline{K}$ by pole placement for the observation error:</u>

Transforming of the estimating error differential equation into the frequency domain.

$$s\tilde{\underline{X}}(s) = (\underline{A} - \underline{K}\,\underline{C}^T)\,\tilde{\underline{X}}(s) \qquad \Rightarrow (s\underline{I} - \underline{A} + \underline{K}\,\underline{C}^T)\,\tilde{\underline{X}}(s) = 0$$

$$\rightsquigarrow \tilde{\underline{X}}(s) = (s\underline{I} - \underline{A} + \underline{K}\,\underline{C}^T)^{-1} \qquad \therefore \underline{X}(s) = (s\underline{I} - \underline{A} + \underline{K}\,\underline{C}^T)^{-1} \cdot 0$$

The Eigenvalues (poles) for the estimation error result from the characteristic equation

$$\det(s\underline{I} - \underline{A} + \underline{K}\,\underline{C}^T) = 0$$

Equivalent approach for the characteristic eqns

$$(s - \lambda_1^B)(s - \lambda_2^B) \cdots (s - \lambda_n^B) = 0$$

Setting equal these two approaches results in

$$\det(s\underline{I} - \underline{A} + \underline{K}\,\underline{C}^T) = (s - \lambda_1^B)(s - \lambda_2^B) \cdots (s - \lambda_n^B)$$

The comparison of co-efficients results in a system of n equations, that can be resolved after the n elements of the observation vector $\underline{K}$. $\lambda_1^B, \lambda_2^B \ldots \ldots \lambda_n^B$ are the given Eigenvalues for the estimation error differential equation of the observer. It is expedient to choose exclusively eigenvalues, which are located on the negative real axis in the complex s plane.

Procedure for the concrete application

- Step 1 :
  Calculation of the left side of the equation by inserting the elements for $\underline{A}$ and $\underline{C}^T$.

The unknown parameter for $\underline{K} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix}$ remain inserted in general form.

- Step 2 :

  Calculation of the right side of the equation by the determination of the Eigenvalues (poles) $\lambda_1^B, \lambda_2^B, \dots \lambda_n^B$

- Step 3 :

  Comparison of the co-efficients for both sides of the equation regarding all powers of the complex frequency s

- Step 4 :

  solving n equations according to n unknowns. Then, the observation vector $\underline{K}$ is found.

Prerequisite for the design of an observer is the observability of the system.

Example :

For the following system $\underline{K}$ should be found :

$$\dot{x}(t) = \begin{bmatrix} -3 & 1 \\ 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} x(t)$$

Determination of the observability

$$\underline{Q}_0 = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T \underline{A} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{6}{3} & -\frac{2}{3} \end{bmatrix} \rightsquigarrow \text{rank}\, \underline{Q}_0 = 2$$

$\rightsquigarrow$ the system is observable

an observer can be designed.

Task : The Eigenvalues of the estimation error differential equation shall be placed at $\lambda_1^B = -10$, $\lambda_2^B = -20$.

$$\det(s\underline{I} - \underline{A} + \underline{K}\,\underline{c}^T) = (s-\lambda_1^B)\cdot(s-\lambda_2^B)$$

1. left side of the equation with $\underline{K} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$

$$\det(s\underline{I} - \underline{A} + \underline{K}\,\underline{c}^T) = \det \begin{bmatrix} s+3+\frac{2}{3}k_1 & -1+\frac{1}{3}k_1 \\ \frac{2}{3}k_2 & s+4+\frac{1}{3}k_2 \end{bmatrix}$$

$= s^2 + \left(\frac{2}{3}k_1 + \frac{1}{3}k_2 + 7\right)s + \frac{8}{3}k_1 + \frac{5}{3}k_2 + 12$

2. right side of the equation

$$(s-\lambda_1^B)(s-\lambda_2^B) = (s+10)(s+20)$$

$$= s^2 + 30s + 200$$

3. Comparison of co-efficients for both sides of the eq^n.

s^2 : $1 = 1$

s : $\frac{2}{3}k_1 + \frac{1}{3}k_2 + 7 = 30 \rightarrow k_1 = \frac{69 - k_2}{2}$

1 : $\frac{8}{3}k_1 + \frac{5}{3}k_2 + 12 = 200$

4. solving the Eq^{n}s :

$$\frac{8}{3}\cdot\frac{69-k_2}{2} + \frac{5}{3}k_2 + 12 = 0$$

$\rightarrow k_2 = 288$

$k_1 = (69-288)/2 = -\frac{219}{2} \rightarrow \underline{k} = \begin{bmatrix} -\frac{219}{2} \\ 288 \end{bmatrix}$

Special case: The system already exists in the observability canonical form

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & & -a_1 \\ 0 & 1 & & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots\ 1 & -a_{n-1} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \underline{x}(t)$$

Introduction of an Observer

$$\dot{\hat{\underline{x}}}(t) = \underline{A}\,\hat{\underline{x}}(t) + \underline{b}\,u(t) + \underline{k}\cdot(y(t) - \hat{y}(t))$$

with $\underline{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$

Estimation error differential equation

$$\dot{\tilde{\underline{x}}}(t) = \underbrace{(\underline{A} - \underline{k}\,\underline{c}^T)}_{\tilde{\underline{A}}}\,\tilde{\underline{x}}(t)$$

The evolution matrix for the estimation error results in

$$\tilde{\underline{A}} = \underline{A} - \underline{k}\,\underline{c}^T$$

$$= \begin{bmatrix} 0 & \cdots & & -a_0 \\ 1 & 0 & & -a_1 \\ 0 & 1 & \cdots & -a_2 \\ \vdots & & \ddots & \\ 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} [0, 0 \ldots 1]$$

$$= \begin{bmatrix} 0 & 0 & \cdots & (-a_0 - k_1) \\ 1 & 0 & & (-a_1 - k_2) \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 1 & (-a_{n-1} - k_n) \end{bmatrix}$$

It is valid:

$$a^K_{B,0} = a_0 + k_1$$

$$a^K_{B,1} = a_1 + k_2$$

$$\vdots$$

$$a^K_{B,n-1} = a_{n-1} + k_n$$

The characteristic polynomial can be determined from this as follows

$$s^n + a^K_{B,n-1} s^{n-1} + \ldots + a^K_{B,1} s + a^K_{B,0} = 0$$

Herewith, pole placement can be carried out by the comparison of co-efficients.

Next

- Eigen values
- Non linear Systems

Lec-11

14/01/2022

Oleg Bauer
H-F 103

EX-4

Task1. $F(s) = \frac{(s^2+3s+2)(s+4)}{(s+1)^2(s+3)^2}$

$= \frac{(s+1)(s+2)(s+4)}{(s+1)^2(s+3)^2}$

$= \frac{(s+2)(s+4)}{(s+1)(s+3)^2}$

Numerator and denominator of F(s) now have no common divisor.
Double poles at $s=-3$

Partial fraction decomposition.

$$F(s) = \frac{A}{s+1} + \frac{B_1}{s+3} + \frac{B_2}{(s+3)^2} = \frac{A}{s+1}\cdot\frac{(s+3)^2}{(s+3)^2} + \frac{B_1}{s+3}\cdot\frac{(s+1)(s+3)}{(s+1)(s+3)} + \frac{B_2}{(s+3)^2}\cdot\frac{s+1}{s+1}$$

$$= \frac{As^2+6As+9A+B_1s^2+4B_1s+3B_1+B_2s+B_2}{(s+1)(s+3)^2}$$

$$= \frac{(A+B_1)s^2+(6A+4B_1+B_2)\cdot s+(9A+3B_1+B_2)}{(s+1)(s+3)^2}$$

Co-efficients in comparison of eqns.: $= \frac{s^2+6s+8}{(s+1)(s+3)^2}$

s^2: $A+B_1=1 \Rightarrow A = 1-B_1$

s: $6A+4B_1+B_2 = 6 \rightarrow -2B_1+B_2=0$

$\therefore B_2=2B_1$

s^0: $9A+3B_1+B_2=8$

$\Rightarrow 9-9B_1+3B_1+2B_1=8$

$B_1 = \frac{1}{4}$, $B_2 = \frac{1}{2}$, $A = \frac{3}{4}$

from, $Y(s) = F(s)\cdot U(s)$ it follows,

$$Y(s) = \frac{3}{4}\cdot\underbrace{\frac{1}{s+1}\cdot U(s)}_{X_1(s)} + \frac{1}{4}\cdot\underbrace{\frac{1}{(s+3)}\cdot U(s)}_{X_2(s)} + \frac{1}{2}\cdot\underbrace{\frac{1}{(s+3)^2}\cdot U(s)}_{X_3(s)}$$

Introduction of state variables

$$sX_1(s) = -X_1(s) + U(s)$$
$$sX_2(s) = -3X_2(s) + U(s)$$
$$sX_3(s) = -3X_3(s) + X_2(s)$$

$$X_3(s) = \frac{1}{s+3} \cdot \frac{1}{s+3} \cdot U(s)$$
$$X_3(s) = \frac{1}{s+3} \cdot X_2(s)$$

Inverse Laplace Transformation.

$$\dot{\underline{x}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & -3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \underline{x}(t)$$

Task #2.

Transformation matrix,

$$T = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T^{-1} = \frac{\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}}{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\underline{\bar{A}} = \underline{T} \cdot \underline{A} \cdot \underline{T}^{-1}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cancel{-1} & \cancel{0} & \cancel{0} \\ \cancel{0} & 2 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & -12 \\ 3 & 4 \end{bmatrix}$$

$$\underline{\bar{B}} = \underline{T} \cdot \underline{B} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\underline{\bar{C}} = \underline{C}T^{-1} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\boxed{\underline{\bar{D}} = \underline{\bar{\bar{D}}} = 0} \quad \underline{d} = \underline{\bar{d}} = 0$$

$$\dot{\bar{x}}(t) = \begin{bmatrix} 8 & -12 \\ 3 & -4 \end{bmatrix} \cdot \bar{x}(t) + \begin{bmatrix} 4 \\ 1 \end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \bar{x}(t)$$

b) $\bar{x}(t) = T\,\underline{x}(t)$

$$\bar{x}_1(t) = 2x_1(t) + x_2(t)$$

$$\bar{x}_2(t) = x_1(t)$$

c) Calculation of the complex transfer functions

original System

$$F(s) = \underline{c}^T(s\underline{I} - A)^{-1}\underline{b} + d$$

$$= \frac{\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} s-2 & -3 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{(s-2)(s-2)}$$

$$= \frac{\begin{bmatrix} 3s-6 & 9+s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{s^2-4s+4}$$

$$= \frac{3s-6+18+2s-4}{s^2-4s+4} = \frac{5s+8}{s^2-4s+4}$$

Transformed System:

$$\bar{F}(s) = \bar{\underline{c}}^T(s\underline{I} - \bar{\underline{A}})^{-1}\bar{\underline{b}} + \bar{d}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s-8 & 12 \\ -3 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+4 & -12 \\ 3 & s-8 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}}{s^2-4s-32+36}$$

$$= \frac{\begin{bmatrix} s+7 & s-20 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}}{s^2-4s+4}$$

$$= \frac{4s+28+s-20}{s^2-4s+4} = \frac{5s+8}{s^2-4s+4}.$$

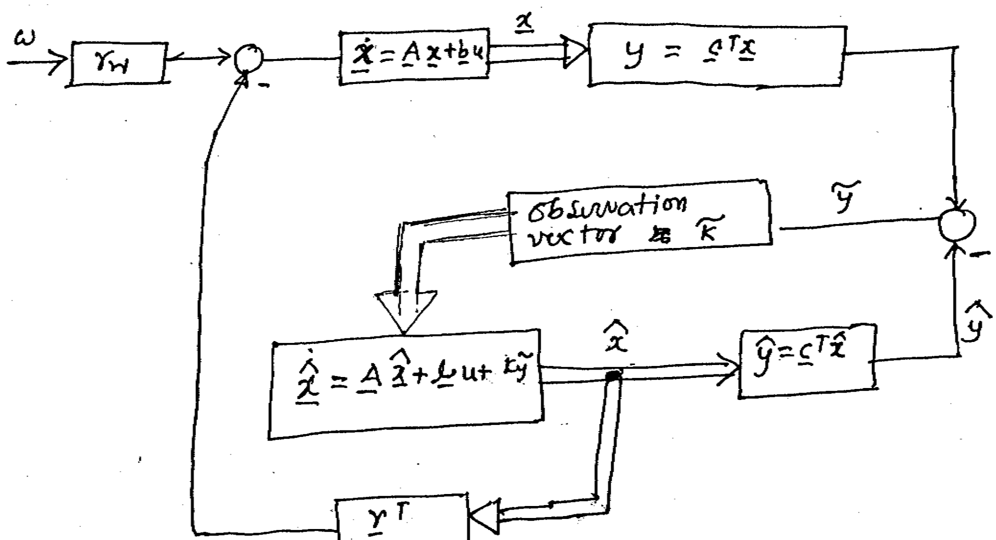
$\therefore \quad F(s) = \bar{F}(s)$

$\therefore$ both complex transfer functions, which describes the input/output behavior of the equivalent (transformed) systems are identical.

Consequence of the Equivalent system transformation:

	transformation Variant	transformation Invariant
A, B, C	X	
D		X
input/output behavior		X
internal system behavior	X	
u(t), y(t)		X
x(t)	X	
controllability, observability		X
Eigen values		X
Eigen vectors	X	

The observer in the closed loop



state feedback control

Theorem of separation:

The Eigen values of the closed loop without the observer are not displaced by the insertion of the observer. The Eigen values of the observer are just added to them.

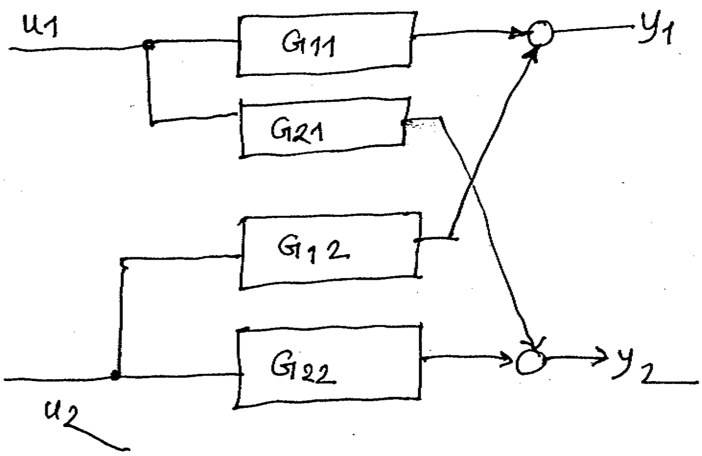
Procedure:

One develops first the automatic controller without regard of the observer. Then the observer is to be designed inserted in to the closed loop.

The decoupling of multivariable systems

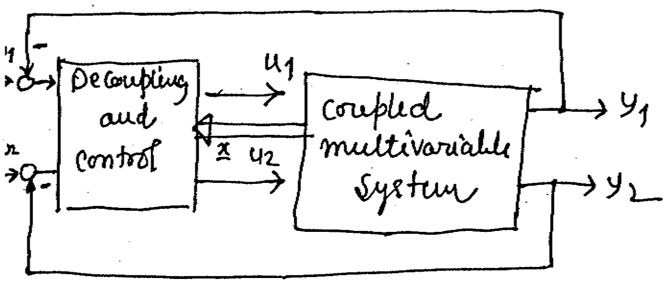
starting point: multivariable systems e.g. with 2 inputs and 2 output variables.

Task: A suitable control system must be found.



Tasks of the control system

* Decoupling of the multivariable systems
* Pole placement for the closed loop
* Comparison between the desired and the controlled variables.



The multivariable system is given in general by

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t)$$

The controlled variables are represented by the output variables $y_1(t)$, $y_2(t) \cdots y_q(t)$ of the system.

For the ~~out~~ i-th output variables it applies $\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$

$$y_i(t) = \underline{c}_i^T \underline{x}(t) + \underline{d}_i^T \underline{u}(t)$$

$\underline{c}_i^T$ and $\underline{d}_i^T$ are the i-th lines of $\underline{C}$ and $\underline{D}$ each

Differential Order:

The differential order δ_i determines, on the which derivative of the output variable ~~of the outputs variable~~ y_i for the first time an ~~output~~ input variable affects. Thereby, it is unimportant which input $u_1(t), u_2(t) \cdots u_q(t)$ acts on it.

The following is valid:

If $\delta_i = 0$; $y_i(t) = \underline{c}_i^T \underline{x}(t) + \underline{d}_i^T \underline{u}(t) \rightarrow \underline{d}_i^T \neq \underline{0}^T$

$\delta_i = 1$
$$y_i(t) = \underline{c}_i^T \underline{x}(t)$$
$$\dot{y}_i(t) = \underline{c}_i^T \dot{\underline{x}}(t)$$
$$= \underline{c}_i^T (\underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t))$$
$$\rightarrow \underline{c}_i^T \underline{B} \neq \underline{0}^T$$

$\delta_i = 2$:
$$y_i(t) = \underline{c}_i^T \cdot \underline{x}(t)$$
$$\dot{y}(t) = \underline{c}_i^T \cdot \underline{A}\,\dot{\underline{x}}(t)$$
$$\dot{y}(t) = \underline{c}_i^T \underline{A} (\underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t))$$

From the evaluation of it follows,

$\delta_i = 0$: $\underline{d}_i^T \neq \underline{0}^T$

$\delta_i = 1$: $\underline{d}_i^T = \underline{0}^T$ $\underline{c}_i^T \underline{B} \neq 0^T$

$\delta_i = 2$: $\underline{d}_i^T = \underline{0}^T$, $\underline{c}_i^T B = \underline{0}^T$, $c_i^T A \cdot B \neq 0$

$\delta_i = 3$: $\underline{d}_i^T = \underline{0}^T$, $\ldots\ldots$ $c_i^T A \cdot B = 0^T$, $\underline{c}_i^T \underline{A}^2 \cdot \underline{B} \neq 0^T$

$\delta_i = 4$: $\underline{d}_i^T = 0 \ldots$ $\underline{c}_i^T \cdot \underline{A}^2 \underline{B} = 0^T$, $\underline{c}_i^T \underline{A}^3 \underline{B} \neq \underline{0}^T$

কতার

i.e. $\delta_i =$ min k, to which applies,

$$\underline{c}_i^T \underline{A}^{k-1} \cdot B \neq \underline{0}^T$$

However if $\underline{d}_i^T \neq \underline{0}^T$ then

$\delta_i = 0$ is valid

Consequently it applies

$$y^{(\delta_i)}(t) = \underline{c}_i x(t) + \underline{d}_i^T u(t) \quad \text{for } \underline{d}_i^T \neq \underline{0}^T$$

Power

$$y_i^{(\delta_i)}(t) = \underbrace{\underline{c}_i^T \cdot \underline{A}^{\delta_i}}_{\underline{c}_i^{*T}} x(t) + \underbrace{\underline{c}_i^T \underline{A}^{\delta_i - 1} \cdot \underline{B}}_{\underline{d}_i^{*T}} \underline{u}(t) \quad \text{for } \underline{d}_i^T = 0$$

Summarized, it applies

$$\underline{y}^*(t) = \underline{c}^* x(t) + \underline{D}^* \underline{u}(t)$$

where

$$c^* = \begin{bmatrix} c_1^{*T} \\ \vdots \\ c_q^{*T} \end{bmatrix} \text{ and } \underline{D}^* = \begin{bmatrix} d_1^{*T} \\ \vdots \\ d_q^{*T} \end{bmatrix}$$

The elements of $y^*(t)$

$$\underline{y}^*(t) = \begin{bmatrix} y_1^*(t) \\ \vdots \\ y_q^*(t) \end{bmatrix} = \begin{bmatrix} y_1^{(\delta_1)} \\ \vdots \\ y_q^{(\delta_q)} \end{bmatrix}$$

contain those derivatives of the output variables $y_1(t) \ldots y_q(t)$, on which for the first time directly an input variable $u_1(t) \ldots u_q(t)$ affects.

<u>Design of the decoupling Control system</u>

The input/output of the system is described by $y^*(t) = \underline{C}^* x(t) + \underline{D}^* \underline{u}(t)$

Approach for the control algorithm,

$$u(t) = \underline{D}^{*-1}\left(-\left(\underline{C}^{*} + \underline{M}^{*}\right) \underline{x}(t) + \underline{L}\,\underline{w}(t)\right)$$

Prerequisite for decoupleability,

$\longrightarrow \mathcal{D}^{*}$ invertible

$\Rightarrow \det(\mathcal{D}^{*}) \neq 0$

one receives for the overall dynamics of the closed loop

$$\underline{y}^{*}(t) = \underline{C}^{*}\underline{x}(t) + \underline{D}^{*}\,D^{*-1}\left(\left(-\left(\underline{C}^{*} + \underline{M}^{*}\right)\right)\underline{x}(t) + \underline{L}\,\underline{w}(t)\right)$$

$$\underline{y}^{*}(t) = -\underline{M}^{*}\underline{x}(t) + \underline{L}\,\underline{w}(t).$$

Conclusion: With decoupling control system a new dynamics can be impressed to the closed loop.
Therefore, the original system dynamics is compensated.

Calculation of the matrices $\underline{M}^*$ & $\underline{L}$

starting point: Desired dynamics for the i-th isolated subsystem

$$y_i^{(\delta_i)}(t) = -\alpha_{i,\delta_i-1}\, y_i^{(\delta_i-1)}(t) - \cdots - \alpha_{i1}\dot{y}_i(t) - \alpha_{i0} y_i(t) + l_i w_i(t)$$

with α_{ij} and l_i constants co-efficients of the differential equation

Normally it is

$$\left.\begin{aligned} &\alpha_{i0} = l_i \text{ controller gain} \\ &\alpha_{i1} \\ &\vdots \\ &\alpha_{i,\delta_i-1}\end{aligned}\right\} \text{determination of the Eigenvalues of the differential eqn}$$

For the evolution of the i-th line $\underline{M}_i^{*T}\underline{x}(t)$ of $\underline{M}^*\underline{x}(t)$ it is valid

$$\underline{M}_i^{*T}\underline{x}(t) = \begin{cases} 0 & \text{for } \delta_i = 0 \\ \sum_{k=0}^{\delta_i-1} \alpha_{ik}\cdot y^{(k)}(t) & \text{for } \delta_i > 0 \end{cases}$$

With the derivatives of y_i it results

$$y_i(t) = \underline{c}_i^T \underline{x}(t)$$
$$\dot{y}_i(t) = \underline{c}_i^T \underline{A}\, \underline{x}(t)$$
$$\ddot{y}_i(t) = \underline{c}_i^T \underline{A}^{\nu}\, \underline{x}(t)$$
$$y_i^{(\delta_i-1)}(t) = \underline{c}_i^T \underline{A}^{\delta_i-1} \underline{x}(t)$$

$$\underline{M}_i^{*T}\underline{x}(t) = \begin{cases} 0 & \text{for } \delta_i = 0 \\ \sum_{k=0}^{\delta_i-1} \alpha_{ik}\, \underline{c}_i^T \underline{A}^k \underline{x}(t) & \text{for } \delta_i > 0 \end{cases}$$

From this one receives the conditional equation for $\underline{M}_i^{*T}$

$$\underline{M}_i^{*T} = \begin{cases} \underline{0}^T & \text{for } \delta_i = 0 \\ \sum_{k=0}^{\delta_i-1} \alpha_{ik}\, \underline{c}_i^T \underline{A}^k & \text{for } \delta_i > 0 \end{cases}$$

Determination of $\underline{L}$:

$$l_i = \alpha_{i0}$$

$$\underline{L} = \begin{bmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_q \end{bmatrix}$$

The transfer matrix of overall system

$$\underline{G}(s) = \begin{bmatrix} G_1(s) & & 0 \\ \vdots & \ddots & \\ 0 & & G_q(s) \end{bmatrix}$$

Herewith, $G_i(s)$, $i = 1, \dots, q$ results in

$$G_i(s) = \frac{Y_i(s)}{W_i(s)}$$

$$G_i(s) = \begin{cases} l_i & \text{for } \delta_i = 0 \\ \dfrac{l_i}{s^{\delta_i} + \alpha_{i,\delta_i-1} s^{\delta_i-1} + \dots + \alpha_{i1} s + \alpha_{i0}} & \text{for } \delta_i > 0 \end{cases}$$

Procedure for the design of the decoupling control system:

- Step 1:

Determine the differential order for every sub system. (Every subsystem ~~comprises~~ precisely with one output variable each)

$$\delta_i = \begin{cases} 0, & \text{if } \underline{d}_i^T \neq \underline{0}^T \\ K, & \text{if } K \text{ is the smallest number in the range } 0 < K \leq n, \text{ for which the following is valid: } \underline{c}_i^T \underline{A}^{K-1} \underline{B} \neq \underline{0}^T \end{cases}$$

- Step 2:

Calculate $\underline{C}^*$ and $\underline{D}^*$

$$\underline{C}^* = \begin{bmatrix} \underline{c}_1^{*T} \\ \vdots \\ \underline{c}_q^{*T} \end{bmatrix}, \quad \underline{c}_i^{*T} = \underline{c}_i^T \underline{A}^{\delta_i}$$

$$\underline{D}^* = \begin{bmatrix} \underline{d}_1^{*T} \\ \vdots \\ \underline{d}_q^{*T} \end{bmatrix}, \quad \underline{d}_i^{*T} = \begin{cases} \underline{d}_i^T & \text{for } \underline{d}_i^T \neq \underline{0}^T \text{ (means, } \delta_i = 0) \\ \underline{c}_i^T \underline{A}^{\delta_i - 1} \underline{B} & \text{for } \underline{d}_i^T = \underline{0}^T \end{cases}$$

- Step 3:

Calculate the matrix $\underline{D}^{*-1}$ by inverting $\underline{D}^*$

The system is only decoupoulable if

$$\det(\underline{D}^*) \neq 0$$

- Step 4:

Determination of the coefficients

$$\begin{matrix} l_1, \alpha_{10}, \dots \alpha_{1,\delta_1 - 1} & \text{with } l_1 = \alpha_{10} \\ \vdots \quad\quad \vdots & \\ l_q, \alpha_{q0}, \dots \alpha_{q,\delta_q - 1} & \text{with } l_q = \alpha_{q0} \end{matrix}$$

due to the desired dynamics for the overall system.

- Step 5:

Setup the matrices $\underline{L}$ & $\underline{M}^*$

$$\underline{L} = \begin{bmatrix} l_1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & & l_q \end{bmatrix}$$

$$\underline{M}^* = \begin{bmatrix} \underline{M}_1^{*T} \\ \vdots \\ \underline{M}_q^{*T} \end{bmatrix}, \quad \underline{M}_i^{*T} = \begin{cases} \underline{0}^T & \text{for } \delta_i = 0 \\ \sum_{k=0}^{\delta_i - 1} \alpha_{ik} \, \underline{c}_i^T \underline{A}^k & \text{for } \delta_i > 0 \end{cases}$$

- Step 6:

Insert the calculated matrices into the control law:

$$\underline{u}(t) = \underline{D}^{*-1}\left(-\underline{c}^* \underline{x}(t) + \underline{L}\,\underline{w}(t) - \underline{M}^* \underline{x}(t)\right)$$

Verbal Explanation:

Every output variable y_i is differentiated as long as one input variable will act in a direct manner on the appropriate derivative.

The resulting system of equations must be solved according to the occuring input variables u_i

The highest derivatives $y^{(\delta_i)}$ are replaced by the desired dynamics for the overall system.

Example:

For the following system, a decoupling controller is to design

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u(t)$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} ; \; y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

1. Differential order:

1. Sub system: $\underline{d}_1^T = \underline{0}^T$; $\underline{c}_1^T \underline{B} = \underline{0}^T$;

$$\underline{c}^T \underline{A}\, \underline{B} = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$\Rightarrow \underline{c}_1^T \underline{A}\, \underline{B} = \begin{bmatrix} 2 & 0 \end{bmatrix} \neq \underline{0}^T$$

$\rightsquigarrow \delta_1 = 2$

2. Sub system: $\underline{d}_2^T = \underline{0}^T$; $\underline{c}_2^T \underline{B} = \begin{bmatrix} 2 & 3 \end{bmatrix} \neq \underline{0}^T$

$\rightsquigarrow \delta_1 = 1$

2. $$\underline{C}^* = \begin{bmatrix} \underline{c}_1^T \underline{A}^2 \\ \underline{c}_2^T \underline{A} \end{bmatrix} = \begin{bmatrix} 0 & 4 & 8 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\underline{D}^* = \begin{bmatrix} \underline{c}_1^T \underline{A}\, \underline{B} \\ \underline{c}_2^T \underline{B} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}$$

3. $D^{*-1} = \dfrac{\begin{bmatrix} 3 & 0 \\ -2 & 2 \end{bmatrix}}{6}$ with $\det(\underline{D}^*) = 6$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

The system is decouplable

4. Determination of the co-efficients

$\underset{\downarrow \text{random}}{\ell_1 = 1} \rightsquigarrow \alpha_{10} = \ell_1 = 1 \qquad \underset{\downarrow \text{random}}{\alpha_{11} = 2}$

$\underset{\downarrow \text{random}}{\ell_2 = 1} \rightsquigarrow \alpha_{20} = \ell_2 = 1$

5. $\underline{L} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underline{M}^* = \begin{bmatrix} \alpha_{10}\, \underline{c}^T + \alpha_{11}\, \underline{c}_1^T \underline{A} \\ \alpha_{20}\, \underline{c}^T \end{bmatrix} = \begin{bmatrix} 2\alpha_{10} & 2\alpha_{11} & 0 \\ 0 & 0 & \alpha_{20} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Control Law:

$$\underline{u}(t) = \underline{D}^{*-1} \left(-\underline{c}^* \underline{x}(t) + \underline{L}\, \underline{\omega}(t) - \underline{M}^* \underline{x}(t) \right)$$

$$= \underline{D}^{*-1} \left(-\begin{bmatrix} 0 & 4 & 8 \\ 1 & 2 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{\omega}(t) - \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(t) \right)$$

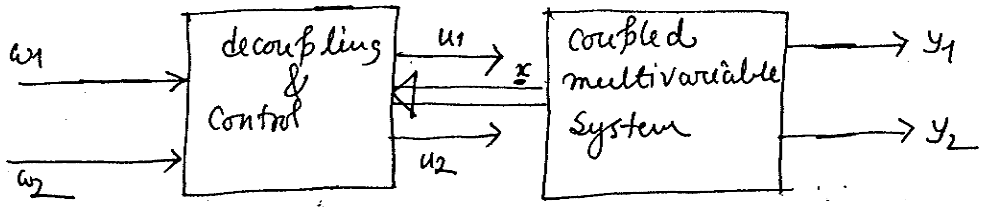
$$= \underline{D}^{*-1} \left(-\begin{bmatrix} 2 & 8 & 8 \\ 1 & 2 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{\omega}(t) \right)$$

$$= D^{*-1} \begin{bmatrix} \omega_1(t) - 2x_1(t) - 8x_2(t) - 8x_3(t) \\ \omega_2(t) - x_1(t) - 2x_2(t) - 2x_3(t) \end{bmatrix}$$

$$u_1(t) = \frac{1}{2}\omega_1(t) - x_1(t) - 4x_2(t) - 4x_3(t)$$

$$u_2(t) = \frac{1}{3}\left(-\omega_1(t) + \omega_2(t) + x_1(t) + 6x_2(t) + 6x_3(t) \right)$$

ok



Test:

$$y_1(t) = 2x_1(t)$$

$$\dot{y}_1(t) = 2\dot{x}_1(t) = 2x_2(t)$$

$$\ddot{y}_1(t) = 2\dot{x}_2(t) = 4x_2(t) + 8x_3(t) + 2u_1(t)$$

$$= 4x_2(t) + \cancel{8x_3(t)} + w_1(t) - 2x_1(t) - 8x_2(t) - \cancel{8x_3(t)}$$

$$= w_1(t) - \underbrace{2x_1(t)}_{y_1} - \underbrace{4x_2(t)}_{2\dot{y}_1}$$

$$\leadsto \ddot{y}_1(t) + \underset{\alpha_{11}}{2}\dot{y}_1(t) + \underset{\alpha_{10}}{1} \cdot y_1(t) = \underset{l_1}{1}\, w_1(t)$$

$$\lambda_1 = \frac{-2 \pm \sqrt{4-4}}{2} = -1, -1$$

aperiodic limit double pole at -1

$$y_2(t) = x_3(t)$$

$$\dot{y}_2(t) = \dot{x}_3(t) = x_1(t) + 2x_2(t) + x_3(t) + 2u_1(t) + 3u_2(t)$$

$$= x_1(t) + 2x_2(t) + x_3(t) + 2\Big(\tfrac{1}{2} w_1(t) - x_1(t) - 4x_2(t) - 4x_3(t)\Big)$$

$$+ 3 \cdot \tfrac{1}{3}\Big(-w_1(t) + w_2(t) + x_1(t) + 6x_2(t) + 6x_3(t)\Big)$$

$$\dot{y}_2(t) = w_2(t) - \underbrace{x_3(t)}_{y_2}$$

$$\leadsto \dot{y}_2(t) + \underset{\alpha_{20}}{1} \cdot y_2(t) = \underset{l_2}{1} \cdot w_2(t)$$

Control of Non. linear System:

linear System:

$$\dot{\underline{x}}(t) = \underline{A}\, \underline{x}(t) + \underline{B}\, \underline{u}(t)$$

$$\underline{y}(t) = \underline{C}\, \underline{x}(t) + \underline{D}\, \underline{u}(t)$$

non linearities:

$\underline{A} \rightarrow \underline{A}(\underline{x}(t))$ $\quad \underline{C} \rightarrow \underline{C}(\underline{x}(t))$

$\underline{B} \rightarrow \underline{B}(\underline{x}(t))$ $\quad \underline{D} \rightarrow \underline{D}(\underline{x}(t))$.

see page 4.

non linear system:

$$\underline{\dot{x}}(t) = \underline{A}(\underline{x}) + \underline{B}(\underline{x}) \cdot \underline{u}(t)$$
$$\underline{y}(t) = \underline{C}(\underline{x}) + \underline{D}(\underline{x}) \cdot \underline{u}(t)$$

Here is the special case that the control signals in $\underline{u}(t)$ are still linear inputs to the system equations.

Example:

$$\dot{x}_1(t) = -x_1(t) \cdot x_2(t) + x_1(t) \cdot u_1(t) + u_2(t)$$
$$\dot{x}_2(t) = -\sqrt{x_1(t)} + \sqrt{x_1(t)} \cdot u_1(t) + (x_1(t) + x_2(t))\, u_2(t)$$

$$y_1(t) = x_1(t) \cdot x_2(t)$$
$$y_2(t) = x_1(t) \cdot u_1(t)$$

⇝ non linear state space description

$$\underline{\dot{x}}(t) = \begin{bmatrix} -x_1(t) \cdot x_2(t) \\ -\sqrt{x_1(t)} \end{bmatrix} + \begin{bmatrix} x_1(t) & 1 \\ \sqrt{x_1(t)} & x_1(t) + x_2(t) \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} x_1(t) \cdot x_2(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_1(t) & 0 \end{bmatrix} \underline{u}(t)$$

shorter, more clear explanation: